Exact Distribution of Random Weighted Convolution of Some Beta Distributions Through an Integral Transform

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Abstract

We give the exact distribution of the average of n independent beta random variables weighted by the selected cuts of $(0,1)$ by the order statistics of a random sample of size $n-1$ from the uniform distribution $U(0,1)$, for each n. A new integral transformation that is similar to generalized Stieltjes transform is given with various properties. Integral representation of the Gauss-hypergeometric function in some parts is employed to achieve the exact distribution. Also the result of Soltani and Roozegar [On distribution of randomly ordered uniform incremental weighted averages: Divided difference approach. Statist Probab Lett. 2012;82(5):1012-1020] with the new transform is achieved. Finally, several new examples of this family of distributions are investigated.

Keywords: Selected order statistics, Beta distribution, Additive Stieltjes transform, Gausshypergeometric function, Random weighted, Exact distribution.

1. Introduction

Van Assche (1987) defined a random variable T uniformly distributed between two independent random variables. He employed the Stieltjes transform and derived that: (i) for X_1 and X_2 on $[-1,1]$, T is uniform on $[-1,1]$ if and only if X_1 and X_2 have an arcsine distribution; and (ii) T have the same distribution as X_1 and X_2 if and only if they are degenerated or have a Cauchy distribution. After that, Johnson and Kotz (1990) showed that the random variable T is a weighted average of X_1 and X_2 with random weights W and $1-W$; $T = WX_1 + (1-W)X_2$, and W is uniform on [0,1] independent of X_1 and X_2. They also extended the randomly weighted averages to more than two random variables with Dirichlet (1,1,...,1) random weights. In fact they neither proved nor disapproved Van Assche's results. Moreover, Johnson and Kotz (1990) discuss on calculating distribution of randomly weighted averages of two independent random variables ($n=2$) not for general n. Also, they cannot find the distribution of randomly weighted averages for general n ($n>2$) and there is not any example there for general n. Soltani and Homei (2009) followed Johnson and Kotz (1990) and investigated Van Assche's results. Soltani and Roozegar (2012) employ certain techniques in divided differences to relate the generalized Stieltjes transform of the distribution of a randomly weighted average of independent random variables $X_1,...,X_m$ to the generalized Stieltjes transforms of the distribution functions $F_1,...,F_m; X_i \sim F_i$, $i=1,...,m$. The random weights are assumed to be cuts of [0,1] by $m-1$ (selected) ordered statistics of independent and identically uniformly distributed random variables $U_1,\ldots,U_n$ on [0,1]; $m \leq n$. Soltani and Homei (2009) treated the case $m=n$ using the Schwartz distribution theory. In this paper, we find the distribution of randomly weighted averages for general n and for Dirichlet $(r_1,r_2,\ldots,r_k)$ weights with a new transform (similar to the Stieltjes transform). Finding examples is difficult, but we identified fairly large classes of randomly weighted average of common beta distributions with Dirichlet $(r_1,r_2,\ldots,r_k)$ random weights by integral representation of the Gauss-hypergeometric function and the new transform.

Recently, many works have been done on randomly weighted averages. Di Salvo (2008) applied the theory on multiple hypergeometric functions and find the distribution of a weighted convolution of gamma variables, Roozegar and Soltani (2014) determine certain classes of power semicircle distributions, that are randomly weighted average distributions. The brief and direct proof of the main result in Roozegar and Soltani (2014) is given in Roozegar and Soltani (2013). A certain version of randomly weighted averages of two independent and continuous random variables with beta random weights is provided in Roozegar (2014). A relation between the Cauchy-Stieltjes transforms of the distribution functions of randomly weighted averages with Dirichlet random proportions and n independent random variables $X_1,\ldots,X_n$ is found in Roozegar (2015). Roozegar and Soltani (2015) investigate the asymptotic behavior of randomly weighted averages. Demni (2016) provide the generalized Stieltjes transforms of some compactly-supported probability distributions with a lot of examples. In Demni (2016) we can find some resaons for importance of convolution of beta distributions. In other work, Roozegar (2017) find the limiting behavior of randomly weighted averages in the case of symmetric heavy-tailed random variables.

2. Randomly weighted averages; Stieltjes transform

Let $U_{(1)} < U_{(2)} < \ldots < U_{(n-1)}$ be order statistics based on a random sample of size $n-1$ from the uniform distribution $U(0,1)$, $U_{(0)}=0$ and $U_{(n)}=1$. Randomly weighted average (RWA) of independent and continuous random variables $X_1,\ldots,X_n$ with respective distribution functions $F_1,\ldots,F_n$ is defined by

$$S_n = R_1X_1 + R_2X_2 + \cdots + R_nX_n, \quad n \geq 2, \tag{1.1}$$

where the proportions $R_i = U_{(i)} - U_{(i-1)}$, $i=1,\ldots,n-1$ and $R_n = 1-\sum_{i=1}^{n-1} R_i$ are random weights. Suppose that among the above whole order statistics, we select $k-1$ order statistics $U_{(n_1)} < U_{(n_2)} < \cdots < U_{(n_{k-1})}$, where $2 \leq k \leq n$ and $n_0 = 0 < n_1 < n_2 < \cdots < n_k = n$. A general form to S_n will be the RWA of k independent and continuous random variables $X_1,\ldots,X_k$, denoted by $S_{n:n_1,\ldots,n_{k-1}}$, which is given by

$$S_{n:n_1,\ldots,n_{k-1}} = \sum_{j=1}^{k} V_j X_j \tag{1.2}$$

where the random weights V_j are defined by

$$V_j = U_{(n_j)} - U_{(n_{j-1})}, \qquad j=1,2,\ldots,k.$$

In the above we have put the conventions $U_{(n_0)}=0$, $U_{(n_k)}=1$ and $r_j = n_j - n_{j-1},\ j = 1,2,\dots,k$. Then $U_{(n_j)} = \sum_{i=1}^{j} V_i$ and $V_j = \sum_{i=n_{j-1}+1}^{n_j} R_i$. The RWA on the form (1.2) was defined and studied by Soltani and Roozegar (2012). In fact the random vector **V**= ($V_1, V_2, \dots, V_k$) has the Dirichlet distribution, Dir ($r_1, r_2, \dots, r_k$), with the probability density function (pdf)

$$f_V\ (v_1,\dots v_k) = \frac{\Gamma(n)}{\prod_{j=1}^{k}\Gamma(r_j)}\prod_{j=1}^{k-1} v_j^{r_j}\,(1-\sum_{i=1}^{k-1} v_i)$$

at any point in the canonical simplex $\{(v_1,\dots,v_k)|\ v_i \geq 0, i=1,2,\dots,k,\ \sum_{i=1}^{k} v_i = 1\}$ in the $(k-1)$-dimensional real space $\mathbb{R}^{k-1}$ and zero outside.

To state assertions, we introduce beta distributions,$\tilde{b}eta(p,q)$ over $[a,b]$ by the pdf

$$f_{p,q}(x) = \begin{cases} \dfrac{1}{B(p,q)(b-a)^{p+q-1}}(x-a)^{p-1}(b-x)^{q-1} & if\ a<x<b \\ 0 & otherwise. \end{cases}$$

for $p,q>0$ and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is beta function.

Let us denote the usual beta distribution over [0,1] by $beta(p,q)$ with above pdf when $a=0$ and $b=1$. The power semicircle distribution with parameters θ and σ on $(-\sigma,\sigma)$, denoted by $PS(\theta,\sigma)$, is a special case of beta distribution whenever $p=q=\theta+3/2$ $b=\sigma$ and $a=-\sigma$. We recall that $PS(-1,\sigma)$ is arcsine distribution on $(-\sigma,\sigma)$, for $\theta=-1/2$ the power semicircle density is the uniform on $(-\sigma,\sigma)$ and for $\theta=0$, it is the semicircle (Wigner) density on $(-\sigma,\sigma)$.

The ordinary Stieltjes transform (ST) and generalized Stieltjes transform (GST) of a distribution H are respectively defined by

$$\mathbb{S}[H](z) = \int_{\mathbb{R}} \frac{1}{z-x}dH(x),\ \ z\in\mathbb{C}\cap(supp H)^c,$$

and

$$\mathbb{S}[H;\rho](z) = \int_{\mathbb{R}} \frac{1}{(z-x)^{\rho}}dH(x),\ \ z\in\mathbb{C}\cap(supp H)^c,\ \rho>0,$$

where $\mathbb{C}$ is the set of complex numbers, supp H stands for the support of H and ρ is a constant. For more on the ST and GST, see Debnath and Bhatta (2007).

Soltani and Roozegar (2012) effectively apply certain results in divided differences and derive the useful relation between GST of $s_{n:n_1,\dots,n_{k-1}}$ and GSTs of $F_1,\dots,F_n$; more is given there:

$$\mathbb{S}[F_{S_{n:n_1,\dots,n_{k-1}}};n](z) = \prod_{i=1}^{k}\mathbb{S}[F_i;r_i](z),\quad z\in\mathbb{C}\bigcap_{i=1}^{k}(supp F_i)^c. \qquad (1.3)$$

In particular

$$\mathcal{S}[F_{S_n};n](z)=\prod_{i=1}^{n}\mathcal{S}[F_i](z),\quad z\in \mathbb{C}\bigcap_{i=1}^{n}(supp F_i)^c.$$

The Gauss-hypergeometric function $F_D^{(1)}$, is defined by the series

$$F_D^{(1)}(c,a;b;z)=\sum_{n=0}^{\infty}\frac{(c)_n(a)_n}{(b)_n n!}z^n,$$

where $(a)_0=1$ and $(a)_n=a(a+1)(a+2)\cdots(a+n-1),\ n\geq 1,$ denotes the rising factorial. Gauss-hypergeometric function $F_D^{(1)}$ has the Euler's integral representation of the form

$$F_D^{(1)}(c,a;b;z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1\frac{t^{a-1}(1-t)^{b-a-1}}{(1-zt)^c}dt. \tag{1.4}$$

For more details on Gauss-hypergeometric function and its properties, see Abramowitz and Stegun (2012) and Andrews et al. (1999).

There are few examples of known distributions that are RWA distributions of the form (1.1), S_n, and there are no examples, to the best of our knowledge, that are RWA distributions of the form (1.2), $S_{n:n_1,\ldots,n_{k-1}}$. Roozegar and Soltani (2013) used the investigations of Demni (2009) and Kubo et al. (2011) on the connection between ST and GST of two distributions and introduced new classes of power semicircle laws that are RWA S_n distributions. In this paper we introduce new classes of RWA $S_{n:n_1,\ldots,n_{k-1}}$ distributions.

Rest of the paper is organized as follows. In Section 2 we introduce a new transform similar to GST for later analysis. We rewrite the main result of Soltani and Roozegar (2012) based on this new transform as Theorem 2.2 in this section. Two new classes of RWA $S_{n:n_1,\ldots,n_{k-1}}$ distributions are investigated in Section 3. In Section 4, we present some examples of the new classes.

3. New transform and RWA distribution

First, we define a new univariate characteristic function called an additive Stieltjes transform (AST).

Definition 1. *If X is a random variable with distribution H on a subset S of $A=[-a,a]$, $a>0$, its AST is defined as*

$$AS[H{:}d](z)=E\left[\frac{1}{(1-zX)^d}\right]$$
$$=\int\frac{1}{(1-zx)^d}dH(x),\qquad |z|<\frac{1}{a}$$

where d is a positive real number.

The assumptions that d is positive and H has a support in S are needed for the one to one correspondence between AST and H in next theorem.

Although the AST and GST transforms have different domains of definition but analytically they have the following relationship which will be useful for the rest of the paper. Specially this issue is not important for next two family of randomly weighted averages. The distribution H is assumed to produce convergent both integrals as mentioned above for each $z \in \mathbb{C}\hat{a}$ so that the function $\frac{1}{(1-zx)^d}$ is holomorphic in $z \in \mathbb{C} \setminus (-\infty, 0]$. For more information and detailed overview of the properties of GST, see Karp and Prilepkina (2012) and Bryc and Hassairi (2011).

$$A\mathcal{S}[H;d](z) = \frac{1}{z^d} \mathcal{S}[H;d](\frac{1}{z}). \tag{2.1}$$

Theorem 1. *For distributions H_1 and H_2 in a subset S of $A = [-a, a]$ and any positive real number d, if we have*

$$A\mathcal{S}[H_1;d](z) = A\mathcal{S}[H_2;d](z), \tag{2.2}$$

for all $|z| < \hat{a}$, then $H_1 = H_2$.

Proof. Since d is a real number and $|zx| < 1$, we have

$$(1-z)^{-d} = \sum_{m=0}^{\infty} \frac{(d)_m}{m!} (zx)^m \tag{2.3}$$

By Deffnition 2.1 and equations (2.2) and (2.3), we have

$$\sum_{m=0}^{\infty} \frac{(d)_m}{m!} z^m \int_S x^m dH_1(x) = \sum_{m=0}^{\infty} \frac{(d)_m}{m!} z^m \int_S x^m dH_2(x). \tag{2.4}$$

Given an integer k, treating z as variable and equating the corresponding coefficients of $z^m((d)_m \neq 0$, for all m, as d is positive) for each m in the two sums, we obtain

$$\int_S x^m dH_1(x) = \int_S x^m dH_2(x).$$

Hence we see that

$$\int_S P(x)\, dH_1(x) = \int_S P(x)\, dH_2(x),$$

where $P(x)$ is any polynomial function of x, and similarly for any continuous function. Thus $H_1 = H_2$.

The following theorem enables us to represent a relationship between the AST of the distribution of RWA $S_{n:n_1,\ldots,n_{k-1}}$ and to those of $X_1, \ldots, X_k$. This result is the main theorem of Soltani and Roozegar (2012) based on the AST.

Theorem 2. *Let $S_{n:n_1,\ldots,n_{k-1}}$ be the RWA given in (1.2). Assume random variables $X_1,\ldots,X_k$ are independent and continuous with distribution functions $F_1,\ldots,F_k$, respectively. Then*

$$AS[F_{S_{n:n_1,\ldots,n_{k-1}}};\sum_{i=1}^{k} r_i](z)=\prod_{i=1}^{k} AS[F_i;r_i](z),$$

for all $|z|<\frac{1}{a}$.

Proof. Applying Theorem 3.1 of Soltani and Roozegar (2012) and then (2.1) to conclude the result.

4. Families of RWA on some beta distributions

In this section, we provide some families of RWA $S_{n:n_1,\ldots,n_{k-1}}$ distributions.

Theorem 3. *Let $X_i, i=1,2,,\ldots k$ be independent random variables with $\tilde{b}eta(r_i+1/2, r_i+1/2)$ distributions and also let $r_i=n_i-n_{i-1}$. Then the RWA $S_{n:n_1,\ldots,n_{k-1}}$ has $\tilde{b}eta(\sum_{i=1}^{k} r_i+1/2, \sum_{i=1}^{k} r_i+1/2)$ distribution.*

Proof. By definition of AST of beta distribution on $[a,b]$, we have

$$AS[F:r_i](z)=\int_a^b \frac{1}{(1-zx)^{r_i}}\frac{(x-a)^{r_i-1/2}(b-x)^{r_i-1/2}}{B(r_i+\frac{1}{2},\ r_i+\frac{1}{2})(b-a)^{2r_i}}\,dx$$

By the change of variable $t=\frac{2x*(b+a)}{b-a}$, we obtain that

$$AS[F:r_i](z)=\frac{1}{[1-\frac{(a+b)z}{2}]}\int_{-1}^{1}\frac{1}{(1-\frac{z(b-a)}{2-(a+b)^z}t)^{r_i}}\frac{(t+1)^{r_i-\frac{1}{2}}}{2^{2riB(r_i+\frac{1}{2},r_i+\frac{1}{2})}}\,dt.$$

By Table 3 in Kubo et al. (2011), it follows that

$$AS[F:r_i](z)=\frac{1}{[1-\frac{(a+b)z}{2}]}\left[\frac{2}{1+\sqrt{1-\frac{z^2(b-a)}{(2-(a+b)z)^2}}}\right]$$

$$=\left[\frac{4}{2-(a+b)z+2\sqrt{1-(a+b)z+abz^2}}\right]^{r_i}$$

Using Theorem 2. we obtain

$$AS[F;\sum_{i=1}^{k} r_i](z)=\left[\frac{4}{2-(a+b)z+2\sqrt{1-(a+b)z+abz^2}}\right]^{\Sigma r_i}$$

which is the AST of $\tilde{b}eta(\sum_{i=1}^{k} r_i+1/2, \sum_{i=1}^{k} r_i+1/2)$ distribution. Therefore $S_{n:n_1,\ldots,n_{k-1}}$ has $\tilde{b}eta(\sum_{i=1}^{k} r_i+1/2, \sum_{i=1}^{k} r_i+1/2)$ distribution.

Another general family of RWA on beta distributions on $[a,b]$ is presented in the following theorem.

Theorem 4. *Let* $X_i, i = 1,2,...,\mathrm{k}$ *be independent random variables with* $\tilde{b}\mathrm{eta}(s_i, r_i - s_i)$ *distributions and also let* $r_i = n_i - n_{i-1}$. *Then the RWA* $S_{n:n_1,...,n_{k-1}}$ *has* $\tilde{b}eta(\sum_{i=1}^{k} s_i, \sum_{i=1}^{k} r_i - \sum_{\mathrm{i}=1}^{\mathrm{k}} s_i)$ *distribution.*

Proof. By using AST of $\tilde{b}\mathrm{e}ta(s_i, r_i - s_i)$ distribution, we have

$$AS[F{:}r_i\ ](z) = \int_a^b \frac{1}{(1-zx)^{r_i}} \frac{(x-a)^{si-1}(b-x)^{ri-si-1}}{B(s_i,r_i-s_i)(b-a)^{ri-1}}\, dx.$$

Changing of variable implies

$$AS[F;r_i](z) = \frac{1}{(1-za)^{r_i}} \int_0^1 \frac{1}{(1-\frac{z(b-a)}{1-za}t)^{ri}} \frac{t^{si-1}(1-t)^{ri-si-1}}{B(s_i,r_i-s_i)}\, dt.$$

Using (1.4), it follows that

$$AS[F;r_i](z) = \frac{1}{(1-za)^{r_i}} F_D^{(1)}\left(r_i, s_i;\ r_i; \frac{z(b-a)}{1-za}\right) = \frac{1}{(1-za)^{r_i}} \frac{1}{\left(1-\frac{z(b-a)}{1-za}\right)^{s_i}} = \frac{1}{(1-za)^{r_i-s_i}} \frac{1}{(1-zb)^{s_i}}.$$

From Theorem 2, we get

$$AS\left[F_{S_{n:n_1,\dots,n_{k-1}}}; \sum_{i=1}^{k} r_i\right](z) = \frac{1}{(1-za)^{\sum_{i=1}^{k}(r_i-s_i)}} \frac{1}{(1-zb)^{\sum_{i=1}^{k} s_i}}.$$

which is the AST of $\tilde{b}\mathrm{e}ta(\sum_{i=1}^{k} s_i, \sum_{i=1}^{k} r_i - \sum_{\mathrm{i}=1}^{\mathrm{k}} s_i)$ distribution. Hence $S_{n:n_1,...,n_{k-1}}$ has $\tilde{b}eta(\sum_{i=1}^{k} s_i, \sum_{i=1}^{k} r_i - \sum_{\mathrm{i}=1}^{\mathrm{k}} s_{\mathrm{k}})$ distribution.

The following corollary is an immediate consequence of Theorem 4.

Corollary 1. *For independent random variables* $X_i, i = 1,2,\dots,\mathrm{k}$ *with* $\tilde{b}$â菲ta$(r_i/2, r_i/2)$ *distributions, the RWA* $S_{n:n_1,...,n_{k-1}}$ *has* $\tilde{b}\mathrm{eta}(\sum_{i=1}^{k} r_i/2, \sum_{i=1}^{k} r_i/2)$ *distribution.*

The following theorem of Roozegar and Soltani (2014) comes as a corollary to Corollary 1.

Corollary 2. *For every integer n≥2, a power semicircle distribution with shape parameter* $\theta = (n-3)/2$ *and any positive range parameter is a RWA* S_n *distribution.*

Proof. Consider $k = n$, $r_i = 1$ for $i = 1,2,...,\mathrm{k}$, $p = q = 1/2$, $a = -\sigma$ and $b = \sigma$, then use Corollary 3.1 to obtain the result.

5. Examples

Theorem 3 and Theorem 4 in Section 3 give general results on distribution of RWA $S_{n:n_1,\dots,n_{k-1}}$ on some beta distributions on $[a,b]$. In this section we present some special examples of distributions that are randomly weighted average distributions given in (1.2). Other examples of RWA S_n distributions in case of $r_1 = r_2 = \cdots = r_k = 1$ and $k = n$ can be found in Roozegar and Soltani (2014). For the first and second examples, we use the results of Kubo et al. (2011).

Example 1. Let X_1, X_2 and W be independent random variables with $\widetilde{b}eta(1/2,1/2)$, $\widetilde{b}eta(m-1/2,m-1/2)$ and $beta(1,m-1)$ distributions, respectively. Then randomly weighted average $T = WX_1 + (1-W)X_2$ has $\widetilde{b}eta(m-1/2,m-1/2)$ distribution. Its AST can be written as

$$AS[F_T;m](z) = \frac{1}{\sqrt{1-z(a+b)+abz^2}}\left[\frac{4}{2-(a+b)z+2\sqrt{1-(a+b)z+abz^2}}\right]^{m-1}.$$

Example 2. Let X_1, X_2 and W be independent random variables with $\widetilde{b}eta(3/2,1/2)$, $\widetilde{b}eta(m-1/2,m-1/2)$ and $beta(1,m-1)$ distributions, respectively. Then randomly weighted average $T = WX_1 + (1-W)X_2$ has $\widetilde{b}eta(m+1/2,m-1/2)$ distribution. Its AST is given by

$$AS[F_T;m](z) = \frac{2}{\sqrt{1-zab+\sqrt{1-z(a+b)+abz^2}}}\left[\frac{4}{2-(a+b)z+2\sqrt{1-(a+b)z+abz^2}}\right]^{m-1}.$$

Example 3. Let $X_1, X_2, \dots\dots X_k$ be independent and identically distributed (i.i.d) random variables with common $U(a,b)$ distribution and random vector $(V_1,\dots,V_K)$ has $Dir(2,2,\dots 2)$ distribution. Then by Corollary 1, the RWA $S_{n:n_1,\dots,n_{k-1}}$ has $\widetilde{b}eta(k,k)$ distribution.

Example 4. Let $X_1, X_2, \dots\dots, X_k$ be i.i.d random variables with common semicircle (Wigner) distribution on $(-\sigma,\sigma)$ and random vector $(V_1,\dots,V_k)$ has $Dir(3,3,\dots 3)$ distribution. Then by Corollary 1, the RWA $S_{n:n_1,\dots,n_{k-1}}$ has $\widetilde{b}eta(3k/2,3k/2)$ distribution.

6. Conclusions

In this paper, We compute the exact distribution of the weighted average of n independent beta random variables where the weightes are the selected cuts of $(0,1)$ by the order statistics of a random sample of size $n-1$ from the uniform distribution. We provide A new integral transformation with some of its mathematical properties. Integral representation of the Gauss-hypergeometric function in some parts is employed to achieve the exact distribution. Finally we investigate several new examples of this family of distributions.

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References

1. Abramowitz, M., and Stegun, I. A. (Eds.) (2012). Handbook of mathematical functions: with formulas, graphs, and mathematical tables. Courier Dover Publications.
2. Andrews, G. E., Askey, R., and Roy, R. (1999). Special functions. Cambridge University Press.
3. Bryc, W., and Hassairi, A. (2011). One-sided Cauchy Stieltjes kernel families. Journal of Theoretical Probability, 24(2), 577-594.
4. Debnath, L., and Bhatta, D. (2007). Integral transforms and their applications. Chapman and Hall/CRC Press, Boca Raton.
5. Demni, N. (2009). On generalized Cauchy-Stieltjes transforms of some beta distributions. Communications on Stochastic Analysis, 3, 197–210.
6. Demni, N. (2016). Generalized Stieltjes Transforms of Compactly-Supported Probability Distributions: Further Examples. Symmetry, Integrability and Geometry: Methods and Applications, 12 (035), 1-13.
7. Di Salvo, F. (2008). A characterization of the distribution of a weighted sum of gamma variables through multiple hypergeometric functions. Integral Transforms and Special Functions, 19(8), 563–575.
8. Johnson, N. L., and Kotz, S. (1990). Randomly weighted averages: Some aspects and extensions. The American Statistician, 44(3), 245–249.
9. Karp, D., and Prilepkina, E. (2012). Hypergeometric functions as generalized Stieltjes transforms. Journal of Mathematical Analysis and Applications, 393(2), 348-359.
10. Kubo, I., Kuo, H. H., and Namli, S. (2011). MRM-applicable measures for the power function of the second order. Communications on Stochastic Analysis, 5(4), 647–670.
11. Roozegar, R. (2014). Certain family of weighted averages with beta random weights. Journal of Statistics Applications and Probability Letters, 1(2), 19–22.
12. Roozegar, R. (2017). Limiting behavior of randomly weighted averages of symmetric heavy-tailed random variables. Communications in Statistics-Theory and Methods, 46(9), 4539-4544.
13. Roozegar, R. (2017). Randomly weighted average with Dirichlet component proportions. Communications in Statistics-Simulation and Computation, 46(3), 1808-1813.
14. Roozegar, R. and Soltani, A. R. (2013). A brief determination of certain class of power semicircle distribution. Journal of Statistical Research of Iran, 10, 107–111.

15. Roozegar, R. and Soltani, A. R. (2014). Classes of power semicircle laws that are randomly weighted average distributions. Journal of Statistical Computation and Simulation, 84(12), 2636–2643.

16. Roozegar, R. and Soltani, A. R. (2015). On the asymptotic behavior of randomly weighted averages. Statistics and Probability Letters, 96, 269-272.

17. Soltani, A. R., and Homei, H. (2009). Weighted averages with random proportions that are jointly uniformly distributed over the unit simplex. Statistics and Probability Letters, 79(9), 1215–1218.

18. Soltani, A. R., and Roozegar, R. (2012). On distribution of randomly ordered uniform incremental weighted averages: Divided difference approach. Statistics and Probability Letters, 82(5), 1012–1020.

19. Van Assche, W. (1987). A random variable uniformly distributed between two independent random variables. Sankhya: The Indian Journal of Statistics, Series A, 207–211.