Application of Entropy Measures to a Failure Times of Electrical Components Models

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Abstract

In this article, the differential entropy and $\beta -$ entropy for Nakagami- μ distribution is derived. In addition, the differential entropy and $\beta -$entropy for some selected versions of these distributions are obtained. Further, numerical comparisons are assessed to indicate which selection distribution has advantages over the other selection in terms of relative loss in entropy.

Keywords: Nakagami- μ distribution; Selection models; Weighted distribution; Entropies measures.

1. Introduction

It is worth mentioning that the distribution of Nakagami is a flexible life time model of distribution that could present a suitable fit to a number of failure data sets. The theory of reliability and reliability engineering as well made intensive usage of the distribution of Nakagami. Due to the less property of memory of such distribution, it is fit to model the portion "constant hazard rate" and used in the theory of reliability. Moreover, it is so acceptable as it is so simple to increase rates of failure in a model of reliability. As well, the distribution of Nakagami is considered the favorable distribution to ensure the electrical component reliability compared to the distribution of Weibull, Gamma and lognormal (many researchers studied it was in recent times, including (c.f. [Yacoub (1999), Cheng and Beaulieu (2001), Schwartz et al. (2013), Ahmad and Rehman (2015), Brychkov and Savischenko (2015), Ahmad et al. (2016), Gkpinar et al. (2016), Mudasir et al. (2016) and Kousar and Memon (2017).])

In addition, the idea of information-theoretic entropy was presented for the first time by Shannon (1948) and then by Weiner (1949) in Cybernetics. Through out the last 60 years, after Shannon (1948) had his entropy measure, a big number of papers, books and monographs have been issued on its extensions and applications, including but not restricted to "Renyi (1961)", "Havrdaand Charvat (1967)", "Tsallis (1988)", "Kapur (1989)", "Ullah (1996)","Dragomir (2003)", "Cover and Thomas (2006)", "Asadi et al. (2006)" and "Harremoes (2006)". A renowned parametric extension of the entropy of Shannon is β-entropy, that was defined by "Havrda and Charvat (1967)" and then it was studied in details by "Tsallis (1988)". Though β-entropy was presented for the first time by "Havrda and Charvat (1967)" in the context of theory of cybernetics,Tsallis exploited its non-extensive features and used it in a physical setting.Therefore, "Hence β-entropy" is known as entropy of Tsallis. Over the past fewyears, "Plastino and Plastino (1999)", "Tsallis (2002)", "Tsallis and Brigatti (2004)", "Jizba and Arimitsu (2004)", "Suyari (2004)", "Pharwaha and Singh (2009)" and "Herrmann (2009)" were very keen on studying "Tsallis entropy"properties and applications.

The issue of modeling systems and components lifetime in the theory of reliability and survival analyses among other scientific fields is deemed essential. In a number of actual situations, the measurements were not revealed in accordance with the standard data distribution. This could be because of the real fact that the population units have unequal opportunities to be registered by an investigator.

Novel weight distribution of Nakagami distribution by length-biased distribution with order θ is presented firstly in this study. Azzalini (1985) was the first one to present the skew-normal distribution to integrate a shape /skewness parameter to a normal distribution according to a weighted function denoted by $F(\alpha X)$ where $F(.)$ is cumulative distribution function of random variable X and α is a sensitive skewness parameter. As the intensive work was completed to present a skewness parameter to a number of symmetric distributions, for example skew-*t*, skew-Cauchy, skew-Laplace and skew-logistic. Generally, skew-symmetric distributions were detected and many of their properties and inference procedures were discussed; for instance, see "Gupta and Kundu (2007)" and the recent monograph made by "Genton (2004)". In this study, new chosen distribution by usage of the same idea of "Azzalini (1985)" is provided firstly.

The arrangement of the current paper is provided as follows: In Section 2, a number of initial results which will be utilized in following sections is introduced. Section 3 covers the key results of this paper. In this section, we are going to study the differential entropy and the β-entropy for a distribution of Nakagami that has two parameters, and then we tabulate values for some specialized versions. Furthermore, two weighted version of Nakagami distribution, their differential entropy and β-entropy were derived. Then, section 4 is devoted to stochastic ordering, numerical comparisons and graphs assisted in specifying a number of measures.

2. Preliminaries

The random variable X has the distribution of Nakagami-μ if its probability density function is defined as follows:

$$g(x|\mu,\Omega) = 2(\mu/\Omega)^{\mu} x^{2\mu-1} exp(-\mu x^2/\Omega)/\Gamma(\mu), \quad x > 0, \tag{2.1}$$

as μ is a shape parameter, Ω is a second parameter which control spread (scale parameter) and $\mu;\Omega > 0$. When apply of radio channels modeling $\mu \geq 0.5$, also known as the" fading figure" or "fading parameter" is defined as:

$$\mu = \frac{\Omega^2}{E[(X^2-\Omega)^2]}, \quad \mu \geq 0.5,$$

and Ω represents average power of envelope signal and it can be estimated as

$$\Omega = E[X^2].$$

The m^{th} moment of the Nakagami$-\mu$ distribution is given

$$E[X^m] = \frac{\Gamma(\mu + m/2)}{\Gamma(\mu)} \left(\frac{\Omega}{\mu}\right)^{m/2},$$

where $\Gamma(.)$ is the standard Gamma function. The cumulative distribution of a random variable from (2.1) can be obtained as

$$G(x|\mu,\Omega) = \frac{1}{\Gamma(\mu)} \gamma(\mu, \mu x^2/\Omega), \tag{2.2}$$

and it has a survival function shown below:

$$\bar{G}(x|\mu,\Omega) = \frac{\Gamma(\mu) - \gamma(\mu, \mu x^2/\Omega)}{\Gamma(\mu)},$$

where $\gamma(.,.)$ is the incomplete gamma function.

In Telatar (1999), $G(x|\mu,\Omega)$ is shown in terms of an incomplete gamma function which is dependent on the average signal-to-noise ratio $(\mu x^2/\Omega)$ which that we have denoted by *ASNR*. This enables the system to be characterized.

The basic measure of uncertainty for density function gx is differential entropy

$$H_X(g_X) = E[-In\, g_X(X)] = \int_0^\infty g_X(u) \ln \frac{1}{g_X(u)} du\,. \tag{2.3}$$

The non-negative absolutely continuous random variable X differential entropy is also recognized as the measure of Shannon information or called sometimes "dynamic measure of uncertainty". Intuitively speaking the entropy provides the expected uncertainty included in $g_X(.)$ about the predictability of a result of X, see Ebrahimi and Pellery (1995). As well it measures how the distribution spreads on its domain. A high H_X value corresponds to a low concentration of the mass of X probability.

Havrda and Charvat (1967) presented β $-$entropy class as shown below:

$$H_\beta(g) = \begin{cases} \frac{1}{\beta - 1}\left[1 - \int_0^\infty g^\beta(x)dx\right]; & \beta \neq 1, \beta > 0, \\ H_X & \beta = 1 \end{cases}.$$

where β is a non-stochastic constant.

If $H(f)\left(H_\beta(f)\right)$ and $H(g)\left(H_\beta(g)\right)$ are the two corresponding differential entropies (β $-$entropies) of f and g density functions respectively, thus, the relative loss of entropies while using "g "instead of "f " is defined as follow:

$$S_H(g_X) = \frac{H(f) - H(g)}{H(f)}$$

and

$$S_{H_\beta}(g_X) = \frac{H_\beta(f) - H_\beta(g)}{H_\beta(f)}.$$

In the literature many concepts of partial ordering among random variables were taken into consideration. These concepts are handful in terms of modeling for applications of reliability and economics.

Stochastic orders were utilized throughout the past forty years, at an accelerated rate, in several areas of probability and statistics. Such areas including theory of reliability, theory of queuing, survival analysis, biology, economics, insurance, actuarial science, operations research and management science. Consequently, many stochastic orders were completely looked into as part of the literature. In this section, we introduced a number of such orders taken into account in the present dissertation; see "Müller and Stoyan (2002)" and "Shaked and Shanthikumar (1994; 2007)" for an exhaustive monograph on this topic.

As part of this section, we are mentioning stochastic orders that compare the "location" of random variables, which will be studied in the following results. The most common and important orders, considered in this result, are: the order of usual stochastic $\leq_{st}$, the order of hazard $\leq_{hr}$ and the order of likelihood ratio $\leq_{lr}$.

The below definition is key to our work:

Definition 2.1. Let X and Y be two nonnegative and absolutely continuous random variables, with density function "f " and "g", distribution function "F" and "G", survival function "$\overline{F}$" and "$\overline{G}$ ", hazard rate functions "r_F" and "r_G", differential entropy "$\mathrm{H_X(f)}$" and "$H_Y(g)$" and β −entropy "$H_\beta(f)$" and "$H_\beta(g)$" respectively. We state that X is smaller than Y in the:

i) Likelihood ratio ordering (denoted as $X \leq_{lr} Y$ if $f(x)/g(x)$ is decreasing over the union of the supports of X and Y.

ii) Differential entropy ordering (denoted as $X \leq_D Y$ if $H_X(f) \leq H_Y(g)$ over the union of the supports of X and Y.

iii) β-entropy ordering (denoted as $X \leq_\beta Y$ if $H_\beta(f) \leq H_\beta(g)$ over the union of the supports of X and Y.

Definition 2.2. Let X and Y be two nonnegative and absolutely continuous random variables, having density function f_X and g_Y respectively. Then, the cross-entropy is

$$H(f_{\mathrm{X}}, g_Y) = E[-\ln(f_X(x)/g_Y(x))] = \int_0^\infty f_X(u)\ \ln\frac{1}{f_X(u)/g_Y(u)}\,du\,. \qquad (2.5)$$

3. Main Results

3.1 Nakagami-μ distribution

Let X be a random variable with the probability density function (2.1); then by using (2.3) we have

$$H_X\big(g(x|\mu,\Omega)\big) = \int_0^\infty \frac{2(\mu/\Omega)^\mu x^{2\mu-1} exp(-\mu x^2|\Omega)}{\Gamma(\mu)} \ln\frac{\Gamma(\mu)}{2(\mu/\Omega)^\mu x^{2\mu-1} exp(-\mu x^2|\Omega)}\ dx$$

$$= \frac{2\ln\Gamma(\mu)(\mu/\Omega)^\mu}{\Gamma(\mu)}\int_0^\infty x^{2\mu-1}\ exp(-\mu x^2/\Omega)\ dx - \frac{2(\mu/\Omega)^\mu}{\Gamma(\mu)}$$

$$\times\int_0^\infty x^{2\mu-1}\ exp(-\mu x^2/\Omega)\left(ln\,2 + \mu\ln(\mu/\Omega) + (2\mu-1)\ln x - \frac{\mu x^2}{\Omega}\right)dx,$$

since

$$\int_0^\infty x^{2\mu-1}\ exp(-\mu x^2/\Omega)dx = \Gamma(\mu)/\left(2\left(\frac{\mu}{\Omega}\right)^{\mu}\right),$$

and

$$\int_0^\infty x^{2\mu-1}\ exp(-\mu x^2/\Omega)\ln x dx = \frac{1}{4}\Gamma(\mu)\left(\frac{\mu}{\Omega}\right)^{-\mu}[\psi(\mu)-\ln\mu+\ln\Omega],$$

we have

$$H_X\big(g(x|\mu,\Omega)\big) = \mu\big(1-\psi(\mu)\big)+\frac{1}{2}\psi(\mu)+\ln\left(\left(\frac{\Omega}{\mu}\right)^{\frac{1}{2}}\frac{\Gamma(\mu)}{2}\right),$$

where ψ is the digamma function.

As well, β −entropy could be gained by the usage of (2.1) and (2.4) as shown below:

$$\begin{aligned} H_\beta\big(g(x|\mu,\Omega)\big) &= \frac{1}{\beta-1}\left[1-\left(\frac{2(\mu/\Omega)^\mu}{\Gamma(\mu)}\right)^\beta \int_0^\infty \left(x^{2\mu-1}\ exp(-\mu x^2/\Omega)\right)^\beta dx\right] \\ &= \frac{1}{\beta-1}\left[1-\left(\frac{2(\mu/\Omega)^\mu}{\Gamma(\mu)}\right)^\beta \frac{1}{2}\left(\frac{\Omega}{\beta\mu}\right)^{\mu\beta-\frac{\beta}{2}+\frac{1}{2}}\int_0^\infty y^{\mu\beta-\frac{\beta}{2}+\frac{1}{2}}\ exp(-y)dy\right] \\ &= \frac{1}{\beta-1}\left[1-\Gamma(\mu)^{-\beta}\beta^{-\mu\beta+\frac{\beta-1}{2}}\left(\frac{\mu}{\Omega}\right)^{\frac{\beta-1}{2}}\Gamma\left(\mu\beta+\frac{1-\beta}{2}\right)\right]. \end{aligned}$$

Note that

$$\begin{aligned} \lim_{\beta\to 1} H_\beta\big(g(x|\mu,\Omega)\big) &= \mu\big(1-\psi(\mu)\big)+\frac{1}{2}\psi(\mu)+\ln\left(\left(\frac{\Omega}{\mu}\right)^{\frac{1}{2}}\frac{\Gamma(\mu)}{2}\right) \\ &= H_X\big(g(x|\mu,\Omega)\big). \end{aligned}$$

The Nakagami distribution covers a wide range of fading conditions. A special case of the Nakagami distribution in which $\mu = 0.5$ implies the one-sided Gaussian distribution (OSG(Ω)). Also, when $\mu = 1$, it implies the Rayleigh distribution ($RA(\Omega)$). In addition, if Y belongs to gamma distribution ($G(\theta_1,\theta_2)$) with shape and scale parameters θ_1 and θ_2 respectively, then $\sqrt{Y}$ belongs to Nakagami distribution with parameters $\mu = \theta_1$ and $\Omega = \theta_1\theta_2$. Finally, if 2μ is integer-valued and if B follows a chi distribution (Chi(2μ)) with parameters 2μ, hence $\sqrt{(\Omega / 2\mu)B} \in g(x\,|\,\mu,\Omega)$.

Table 1: Differential entropy measure for some particular values of the parameters for Nakagami-μ distributions

Distribution	Density function	Differential entropy
$OSE(\Omega)$	$g(x)=\sqrt{2/\pi}\,\Omega^{-1/2}\exp(-x^2/2\Omega)$	$\ln\sqrt{\pi}(2/\Omega)^{-1/2}+\frac{1}{2}$
$G(\mu,\theta)$	$g(y)=(1/\theta)^\mu y^{\mu-1}\exp(-y/\theta)\Gamma(\mu)$	$\mu(1-\psi(\mu))+\psi(\mu)+\ln(\theta\,\Gamma(\mu))$
$Chi(2\mu)$	$g(y)=2^{(1-\mu)}y^{2\mu-1}\exp\left(-\frac{y^2}{2}\right)/\Gamma(\mu)$	$\mu(1-\psi(\mu))+\frac{1}{2}\psi(\mu)+\ln\left(\sqrt{\frac{1}{2}}\Gamma(\mu)\right)$
$RA(\Omega)$	$g(x)=(2x/\Omega)\exp(-x^2/\Omega)$	$1+\ln(\sqrt{\Omega}/2)+(\gamma/2)$

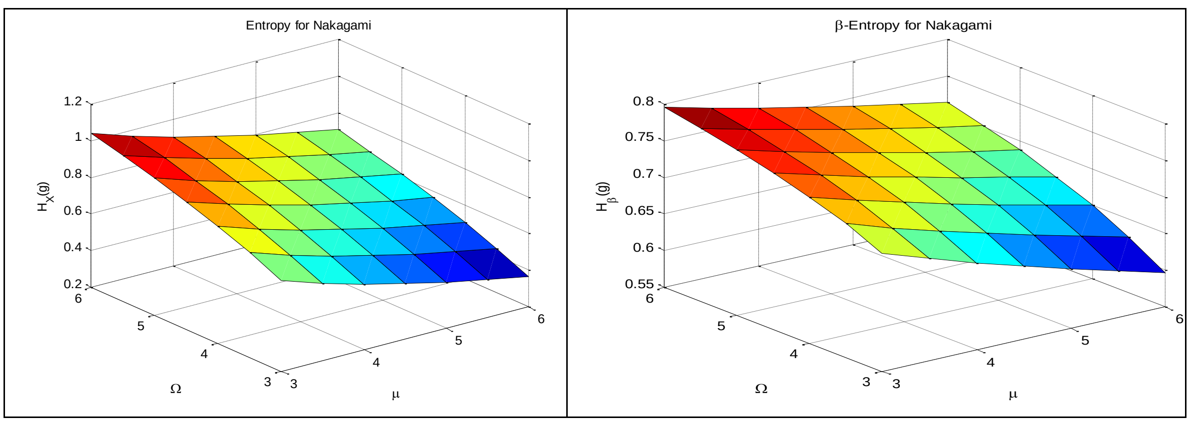
Table 2: $\boldsymbol{\beta}$ −Entropy measures for some particular values of the parameters for Nakagami-μ distributions

Distribution	Density function	β -entropy
$OSE(\Omega)$	$g(x)=\sqrt{2/\pi}\Omega^{-1/2}\exp\left(-x^2/2\Omega\right)$	$\frac{1}{\beta-1}\left(1-\pi^{-\beta-1/2}\sqrt{\beta}\left(2\Omega\right)^{\frac{\beta-1}{2}}\right)$
$G(\mu,\theta)$	$g\left(y\right)=(1/\theta)^{\mu}\, y^{\mu-1}\exp\left(-y/\theta\right)\Gamma\left(\mu\right)$	$\frac{1}{\beta-1}\left(1-\left(1/\theta\right)^{\beta\mu-\mu}/\Gamma^{\beta-1}(\mu)\right)$
$Chi(2\mu)$	$g\left(y\right)=2^{(1-\mu)}y^{2\mu-1}\exp\left(-\frac{y^2}{2}\right)/\Gamma\left(\mu\right)$	$\frac{1}{\beta-1}\left(1-\frac{2^{\frac{\beta-1}{2}}}{\Gamma^{\beta}(\mu)}\left(\frac{1}{\beta}\right)^{\beta\mu-\frac{\beta-1}{2}}\Gamma\left(\beta\mu-\frac{\beta-1}{2}\right)\right)$
$RA(\Omega)$	$g(x)=\left(2x/\Omega\right)\exp\left(-x^2/\Omega\right)$	$\frac{1}{\beta-1}\left(1-2^{\beta-1}\Omega^{\frac{2-\beta}{2}}\beta^{\frac{2\beta-1}{2}}\Gamma\left(\beta-1/2\right)\right)$

where γ is the Euler-Mascheroni constant$\left(\lim_{n\to\infty}\left(-\ln(n)+\sum_{k=1}^{n}\frac{1}{k}\right)\right)$.

The differential entropy and β-entropy values for particular values of the parameters for some versions of the Nakagami-μ distributions have been derived and are summarized in Table 1 and Table 2.

Fig 1: Entropy for Versions of Nakagami Distribution



2.2 Size-biased (SB) Nakagami$-\mu$ distribution of order θ

Let X be a random variable with the probability density function (2.1) and $w(x)=x^{\theta}$as result the weighted distribution function could be expressed as shown below:

$$g^{w\theta}(x)=\frac{x^{\theta}g(x)}{E_{g^{w\theta}}[x^{\theta}]}\;; \qquad x>0, \tag{3.1}$$

the parameter θ in the above weighted distribution is said to the moment parameter. When $\theta=1$, (3.1) is considered as the length-biased distribution.

As

$$\int_0^\infty t^{\upsilon-1} \ln t \, exp(-pt)\, dt = \Gamma(\upsilon)p^{-\upsilon}[\psi(\upsilon) - \ln p]\,. \tag{3.2}$$

Thus,

$$E_{g^{w\theta}}[x^\theta] = \frac{1}{\Gamma(\mu)}\left(\frac{\Omega}{\mu}\right)^{\frac{\theta}{2}} \Gamma\left(\mu + \frac{\theta}{2}\right).$$

By using (3.1), we gain size-biased (SB) nakagami-μ density function of order θ as shown below:

$$g^{w_\theta}(x|\mu,\Omega) = \frac{2(\mu/\Omega)^{\mu+\frac{\theta}{2}} x^{2\mu+\theta-1}\, exp(-\mu x^2/\Omega)}{\Gamma\left(\mu+\frac{\theta}{2}\right)}\,. \tag{3.3}$$

In recent time, Mudasir et al. (2016) presented length biased of nakagami distribution of order 1 (which is special case of (3.2)).

The cumulative distribution of a random variable from (3.3) could be gained as shown below:

$$G^{W_\theta}(x|\mu,\Omega) = \frac{2(\mu/\Omega)^{\mu+\frac{\theta}{2}}}{\Gamma\left(\mu+\frac{\theta}{2}\right)} \int_0^t x^{2\mu+\theta-1}\, exp(-\mu x^2/\Omega)\, dx\,.$$

$$= \frac{\gamma\left(\mu+\frac{\theta}{2}, \mu t^2/\Omega\right)}{\Gamma\left(\mu+\frac{\theta}{2}\right)}\,.$$

Through the usage of Eq. (3.2), integration (3.3) and after some algebra, we get

$$H\left(g^{W_\theta}(x|\mu,\Omega)\right) = \ln 2 + \left(\mu+\frac{\theta}{2}\right)[\ln(\mu/\Omega) - 1] + \frac{2\mu+\theta-1}{2}\left[\psi\left(\mu+\frac{\theta}{2}\right) - \ln\left(\frac{\mu}{\Omega}\right)\right] - \ln\Gamma\left(\mu+\frac{\theta}{2}\right).$$

Through a similar process, β −entropy of weighted Nakagami (N^w) could be gained as shown below:

$$H_\beta\left(g^{W_\theta}(x|\mu,\Omega)\right) = \frac{1}{\beta-1}\left(1 - \frac{2^\beta(\mu/\Omega)^{\beta\left(\mu+\frac{\theta}{2}\right)}}{\Gamma^\beta\left(\mu+\frac{\theta}{2}\right)} \int_0^\infty x^{\beta(2\mu+\theta-1)}\, exp(-\beta\mu x^2/\Omega)\ dx\right)$$

$$= \frac{1}{\beta-1}\left(1 - \frac{2^{\beta-1}(\Omega/\mu)^{\frac{1}{2}(1-\beta)}}{\Gamma^\beta\left(\mu+\frac{\theta}{2}\right)} (1/\beta)^{\beta\mu+\frac{1}{2}(\beta\theta-\beta+1)}\Gamma\left(\beta\mu + \frac{\beta}{2}\left(\theta - \frac{1}{\beta} - 1\right)\right)\right)$$

Table 3: Differenatial Entropy for some particular values of the parameters for size-biased Nakagami$-\mu$ distributions

Distribution	**Density function**	**Differential entropy**
$SB^{\theta}\left(OSE\ (\Omega)\right)$	$\frac{2(1/2\Omega)^{\frac{\theta+1}{2}} x^{\theta}\exp\left(-x^2/2\Omega\right)}{\Gamma(1+\theta/2)}$	$\ln 2-\frac{1+\theta}{2}\left[\ln(2\Omega)+1\right]+\frac{\theta}{2}\left[\psi\left(\frac{1+\theta}{2}\right)+\ln(2\Omega)\right]-\ln\Gamma\left(\frac{1+\theta}{2}\right)$
$SB^{\theta}\left(G(\mu,\theta)\right)$	$\frac{(\mu/\Omega)^{\mu+\theta/2} x^{\mu+\frac{\theta}{2}-1}\exp(-\mu x/\Omega)}{\Gamma(\mu+\theta/2)}$	$\mu+\frac{\theta}{2}-\ln\frac{(\mu/\Omega)^{\frac{2\mu+\theta}{2}}}{\Gamma(\mu+\theta/2)}-(\mu+\theta/2-1)\left(\psi\left(\frac{2\mu+\theta}{2}\right)-\ln(\mu/\Omega)\right)$
$SB^{\theta}\left(Chi(2\mu)\right)$	$\frac{2^{1-\mu+\frac{\theta}{2}} x^{2\mu+\theta-1}\exp\left(-\frac{x^2}{2}\right)}{\Gamma(\mu+\theta/2)}$	$\mu+\frac{\theta}{2}-\ln\frac{2^{1-\mu+\frac{\theta}{2}}}{\Gamma\left(\frac{2\mu+\theta}{2}\right)}-\frac{2\mu+\theta-1}{2}\left(\ln 2+\psi\left(\frac{2\mu+\theta}{2}\right)\right)$
$SB^{\theta}\left(RA(\Omega)\right)$	$\frac{2(1/\Omega)^{\frac{2+\theta}{2}} x^{\theta+1}\exp\left(-x^2/\Omega\right)}{\Gamma(2+\theta/2)}$	$\ln 2-\frac{\theta+2}{2}\left[\ln\Omega+1\right]+\frac{\theta+1}{2}\left[\psi\left(\frac{\theta+2}{2}\right)+\ln\Omega\right]-\ln\Gamma\left(\frac{\theta+2}{2}\right)$

Table 4: β −Entropy measures for some particular values of the parameters for size-biased Nakagami$-\mu$ distributions

Distribution	**Density function**	β **-entropy**
$SB^{\theta}\left(OSE(\Omega)\right)$	$\frac{2(1/2\Omega)^{\frac{\theta+1}{2}} x^{\theta}\exp\left(-x^2/2\Omega\right)}{\Gamma(1+\theta/2)}$	$\frac{1}{\beta-1}\left(1-\frac{2^{\frac{\beta+1}{2}}\Omega^{-\frac{\beta}{2}}\beta^{-\frac{(\beta\theta+1)}{2}}}{\Gamma^{\beta}\left(\frac{\theta+1}{2}\right)\Gamma^{-1}\left(\frac{\beta}{2}\left(\theta-\frac{1}{\beta}\right)\right)}\right)$
$SB^{\theta}\left(G(\mu,\theta)\right)$	$\frac{(\mu/\Omega)^{\mu+\theta/2} x^{\mu+\frac{\theta}{2}-1}\exp(-\mu x/\Omega)}{\Gamma(\mu+\theta/2)}$	$\frac{1}{\beta-1}\left[1-\frac{(\mu/\Omega)^{\beta-1}\beta^{-\beta\left(\mu+\frac{\theta}{2}-1\right)}\left(\mu+\frac{\theta}{2}-1\right)}{\Gamma^{\beta}\left(\mu+\frac{\theta}{2}\right)\Gamma^{-1}\left(\beta\left(\mu+\frac{\theta}{2}-1\right)\right)}\right]$
$SB^{\theta}\left(Chi(2\mu)\right)$	$\frac{2^{1-\mu+\frac{\theta}{2}} x^{2\mu+\theta-1}\exp\left(-\frac{x^2}{2}\right)}{\Gamma(\mu+\theta/2)}$	$\frac{1}{\beta-1}\left(1-\frac{2^{\frac{\beta-1}{2}}\beta^{-\frac{\beta}{2}(2\mu+\theta-1)-\frac{1}{2}}}{\Gamma^{\beta}\left(\mu+\frac{\theta}{2}\right)\Gamma^{-1}\left(\frac{\beta}{2}(2\mu+\theta-1)+\frac{1}{2}\right)}\right)$
$SB^{\theta}\left(RA(\Omega)\right)$	$\frac{2(1/\Omega)^{\frac{2+\theta}{2}} x^{\theta+1}\exp\left(-x^2/\Omega\right)}{\Gamma(2+\theta/2)}$	$\frac{1}{\beta-1}\left(1-\frac{2^{\beta-1}\Omega^{\frac{1-\beta}{2}}\beta^{-\beta-\frac{1}{2}(\beta\theta-\beta+1)}}{\Gamma^{\beta}\left(1+\frac{\theta}{2}\right)\Gamma^{-1}\left(\beta+\frac{\beta}{2}\left(\theta-\frac{1}{\beta}-1\right)\right)}\right)$

where γ is the Euler-Mascheroni constant $\left(\lim_{n\to\infty}\left(-\ln(n)+\sum_{k=1}^{n}\frac{1}{k}\right)\right)$, $SB^{\theta}(\mathrm{OSG}(\Omega))$ is SB of one-sided Gaussian distribution with order θ, $SB^{\theta}(G\ (\mu,\theta))$ is SB of Gamma distribution with order θ, $SB^{\theta}(Chi(2\mu))$ is SB of Chi-distribution with order θ and $SB^{\theta}\ (RA\ (\Omega))$ is SB of Rayleigh distribution with order θ. The differential entropy and β-entropy values for particular values of the parameters for SB nakagami-μ distributions of order θ have been derived and are summarized in Table 3 and Table 4.

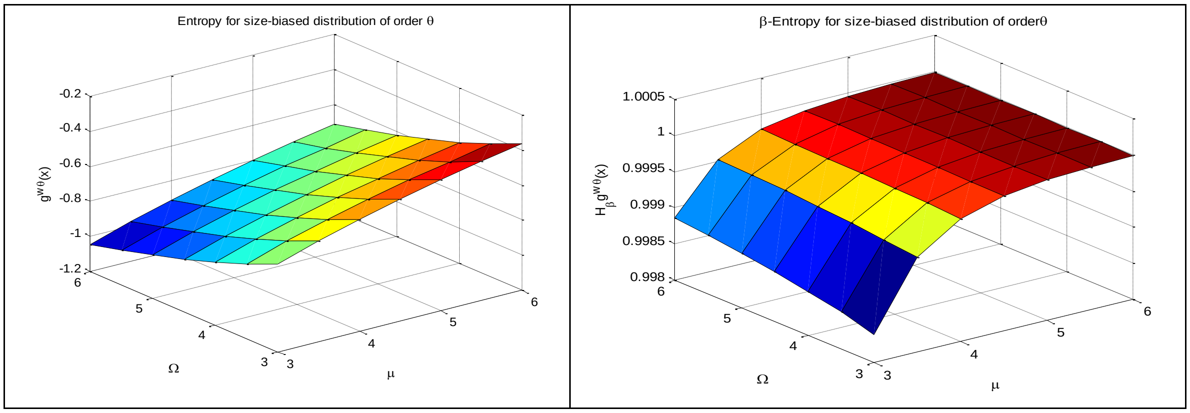
The relative loss of entropies while using size-biased (SB) nakagami-μ distribution of order θ $\left(g^{w\theta}\right)$instead of the nakagami-μ distribution (g) are defined as shown below:

$$S_H\left(g^{W_\theta}(x|\mu,\Omega)\right) = \frac{H(g) - H\left(g^{W_\theta}\right)}{H\left(g(x|\mu,\Omega)\right)},$$

and

$$S_{H_\beta}\left(g^{W_\theta}(x|\mu,\Omega)\right) = \frac{H_\beta\left(g(x|\mu,\Omega)\right) - H_\beta\left(g^{W_\theta}(x|\mu,\Omega)\right)}{H_\beta\left(g(x|\mu,\Omega)\right)}.$$

Fig 2. Entropies for size-biased (SB) Nakagami-μ distribution of order θ=1



3.3 Azzeline weighted version of Nakagami-μ distribution

Surprisingly, though Azzalini's method was utilized intensively for many symmetric and non-symmetric distributions. In this section, it will be noted that if we implement Azzalini's method to the distribution of Nakagami, it will result in a novel class of weighted distributions of Nakagami (WN) with a sensitive skewness parameter. Now we denote a member of this novel class of weighted distributions as WN distribution. This sensitive skewness parameter regulates the shape of the probability density function (pdf) of the WN distribution.

Let X be a non-negative random variable with an absolutely continuous distribution function G, survival function $\overline{G}$ and probability density function (pdf) g. Let $l_X = inf\,\{x \in \mathbb{R}^1 : G(x) > 0\}$, $u_X = sup\,\{x \in \mathbb{R}^1 : F(x) < 1\}$ and $S_X = (l_X, u_X)$. Let $w(.)$ be a non-negative function defined on the real line.

Consider that when $w(x) = G(\alpha x)$, the weight function is an increasing function for all α. It implies that if the pdf of X is unimodal consequently

$$Mode\ (X^W) > Mode\ (X).$$

Furthermore, the weighted distribution function could be expressed as shown below:

$$F^{W}(t) = \int_0^t \frac{G(\alpha x)G(u)\,du}{E[G(\alpha x)]},$$
$$= \frac{1}{E[G(\alpha x)]}\left\{G(\alpha t)G(t) - \int_0^t G^{\backslash}(\alpha u)G(u)\,du\right\},$$
$$= \frac{G(t)\left[G(\alpha t) - M_F[t]\right]}{E[G(\alpha x)]},$$

where $M_F(t) = \int_0^t G^{\backslash}(\alpha u)G(u)du \; and \; G^{\backslash}(\alpha x) = \partial\, G(\alpha x)/\partial x$.

Moreover, the survival function of the weighted distribution could also be expressed as follows:

$$\bar{F}^{W}(t) = \int_t^{\infty} \frac{-G(\alpha u)\bar{G}^{\backslash}(u)du}{E[G(\alpha t)]} = \frac{\bar{G}(t)[G(\alpha t)+U_F(t)]}{E[G(\alpha t)]}.$$

Here, $U_F(t) = \int_t^{\infty} G^{\backslash}(\alpha u)\bar{G}(u)\,du/\bar{G}(t)$. Note that if $w^{\backslash}(t) > 0$, then $\mathrm{M}_F(t) \geq 0$ for all $t \geq 0$. Now, we define

$$B(x) = E\left[G\left((\alpha x)\middle|X \leq x\right)\right] = \frac{\int_0^x G(\alpha t)\, f(t)dt}{G(x)}.$$

Thus, the distribution function can be rewritten as the shown below

$$F^{W}(x) = \frac{B(x)G(x)}{E[G(\alpha x)]}.$$

As well, assume

$$A(x) = E\left[G\left((\alpha x)\middle|X \;\; \geq x\right)\right] = \frac{\int_x^{\infty} G(\alpha u)g(t)dt}{\bar{G}(x)} .$$

As a result, we could write $E[G(\alpha x)]$ as

$$E[G(\alpha x)] = B(x)G(x) + A(x)\bar{G}(x).$$

Since

$$G(\alpha x|\mu,\Omega) = \frac{1}{\Gamma(\mu)}\gamma(\mu,\mu(\alpha x)^2/\Omega),$$

and

$$E[G(\alpha x|\mu,\Omega)] = \frac{(\mu/\Omega)^{\mu}}{\Gamma^2(\mu)}\int_0^{\infty} y^{\mu-1}\, exp(-\mu y|\Omega)\,\gamma(\mu,\mu\alpha^2 y/\Omega)\,dy . \qquad (3.4)$$

Through using generalized hypergeometric series as shown below:

$${}_mF_n(\alpha_1,\ldots,\alpha_m;c_1,\ldots,c_n;d) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k \ldots (\alpha_m)_k}{(c_1)_k \ldots (c_m)_k}\frac{d^k}{k}$$

as $(a)_0 = 1,\ (a)_k = (a)(a+1)\ldots(a+k-1)$ which is Pochhammer's polynomial.

Inserting this expansion in Eq. (3.4) and, after some algebra, we could be writing

$$E[G(\alpha x|\mu,\Omega)] = \frac{\mu^{2\mu-1}\alpha^{2\mu}\Gamma(2\mu)}{(\mu\alpha^2+\mu)^{2\mu}\Gamma^2(\mu)} \quad {}_2F_1\left(1{,}2\mu;\mu+1;\ \frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right).$$

Then, the probability density function of weighted Nakagami-μ distribution is

$$f_N^w(\mathrm{x}|\mu,\Omega;\alpha) = \frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu}x^{2\mu-1}\,exp\left(-\frac{\mu x^2}{\Omega}\right)\gamma(\mu,\mu(\alpha x)^2/\Omega)}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \ (1{,}2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu})}\ ,\quad x>0 \qquad (3.5)$$

and 0 otherwise, Eq. (3.5) is referred to as $WN(\mu,\Omega,\alpha)$. The distribution function of (3.5) could be written as follows

$$F_N^w(\mathrm{x}|\mu,\Omega;\alpha) = \frac{(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha^2)^{\mu+k}}{k!(\mu+k)\left(\frac{\mu}{\Omega}\right)^{\mu}}\gamma(2\mu+k,x^2)}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \ (1{,}2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu})},$$

By series we could rewrite incomplete gamma function as shown below:

$$\gamma(\alpha,x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{\alpha+k}}{k!\,(\alpha+k)}. \qquad (3.6)$$

As the differential entropy for distribution$f_N^w(x|\mu,\Omega;\alpha)$ could derive as shown below:

$$H_X\left(f_N^w(x|\mu,\Omega;\alpha)\right) = -\int_0^{\infty} f_N^w(x|\mu,\Omega;\alpha)\ln f_N^w(x|\mu,\Omega;\alpha)\,dx. \qquad (3.7)$$

$$= -\left[\ln\frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu})}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \left(1{,}2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}\right]$$

$$-\frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu})}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \left(1{,}2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}$$

$$\times\int_0^{\infty}\begin{pmatrix} x^{2\mu-1}\,exp(-\mu x^2/\Omega)\,\gamma(\mu,\mu(\alpha x)^2/\Omega) \\ \times\ln((x^{2\mu-1}\,exp(-\mu x^2/\Omega)\,\gamma(\mu,\mu(\alpha x)^2/\Omega))\end{pmatrix}dx,$$

Now, suppose

$$\boldsymbol{H}_1(\mu,\Omega) = \int_0^{\infty} x^{2\mu-1}\,exp(-\mu x^2/\Omega)\,\gamma(\mu,\mu(\alpha x)^2/\Omega)\ln x\,dx\,;$$

$$\boldsymbol{H}_2(\mu,\Omega) = \int_0^{\infty} x^{2\mu+1}\,exp(-\mu x^2/\Omega)\,\gamma(\mu,\mu(\alpha x)^2/\Omega)dx\,;$$

$$\boldsymbol{H}_3(\mu,\Omega) = \int_0^{\infty} x^{2\mu-1}\,exp(-\mu x^2/\Omega)\,\gamma(\mu,\mu(\alpha x)^2/\Omega)\ln\gamma(\mu,\mu(\alpha x)^2/\Omega)\,dx.$$

Therefore, we can rewrite (3.7) as follows:

$$H_X\left(f_N^W(x|\mu,\Omega;\alpha)\right) = -\ln \frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu})}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)} \tag{3.8}$$

$$-\frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu}}{(\Omega\alpha^2)^{\mu}\Gamma(2\mu)\ {}_2F_1\ \left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}$$

$$\times\left[(2\mu-1)\boldsymbol{H}_1(\mu,\Omega)-(\mu/\Omega)\boldsymbol{H}_2(\mu,\Omega)+\boldsymbol{H}_3(\mu,\Omega)\right].$$

By using Eq. (15), section 6.455 in Gradshteyn and Ryzhik (2014), we can derive:

$$\boldsymbol{H}_2(\mu,\Omega)=\frac{1}{2}\left[\frac{(\mu\alpha^2/\Omega)^{\mu}}{\mu\left(\frac{\mu\alpha^2}{\Omega}+\frac{\mu}{\Omega}\right)^{2\mu+1}}\quad {}_2F_1\left(1,2\mu+1;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)\right], \tag{3.9}$$

where $Re\ \mu(\alpha^2+1)>0, Re\ \mu>0\ , Re\ (2\mu+1)>0$. By using Eq. (3.6), we can rewrite $\boldsymbol{H}_1(\mu,\Omega)$ as following

$$\boldsymbol{H}_1(\mu,\Omega)=\frac{1}{4}\int_0^{\infty} y^{\mu-1}exp(-\mu y/\Omega)\gamma(\mu,\mu\alpha^2 y/\Omega)\ln y\,dy.$$

The last equation could be expressed as follows

$$\boldsymbol{H}_1(\mu,\Omega)=\frac{1}{4}\frac{\Omega}{\mu\alpha^2}\left(\frac{\Omega}{\mu\alpha^2}\right)^{\mu-1}\int_0^{\infty}(z)^{\mu-1}exp\left(-\frac{z}{\alpha^2}\right)\gamma(\mu,z)\left(\ln\frac{z\Omega}{\mu\alpha^2}\right)dz.$$

In addition,

$$\int_0^{\infty} x^{\alpha+k-1}e^{\frac{-x}{\theta}}\ln x\,dx=\theta^{\alpha+k}(\alpha+k-1)!\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\alpha+k-1}-\ln\left(\frac{\gamma}{\theta}\right)\right];\theta>0,x>0,$$

where

$$\gamma=exp(c), c=\lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n}-\ln k\right)=0.5772156649,$$

and c is Euler-Mascheroni constant. We have

$$\boldsymbol{H}_1(\mu,\Omega)=\frac{1}{4}\left(\frac{\Omega}{\mu\alpha^2}\right)^{\mu}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!(\mu+n)}\left[\left[\begin{array}{c}\alpha^{2\alpha+2k}(2\mu+n-1)!\\ \times\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2\mu+n-1}-\ln\left(\frac{\gamma}{\theta^2}\right)\right]\\ +(\alpha^2)^{2\mu+n}\Gamma(2\mu+n)\ln\left(\frac{\Omega}{\mu\alpha^2}\right)\end{array}\right]\right]\right].$$

By using harmonic number, we have

$$\boldsymbol{B}(n,\mu)=\sum_{k=1}^{2\mu+n-1}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2\mu+n-1}.$$

Thus, we get:

$$\boldsymbol{H}_1(\mu,\Omega)=\frac{1}{4}\left(\frac{\Omega}{\mu\alpha^2}\right)^{\mu}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!\,(\mu+n)}\left[\left[\begin{array}{l}\alpha^{2\alpha+2k}(2\mu+n-1)!\\ \times\left(\boldsymbol{B}(n,\mu)-\ln\left(\frac{\gamma}{\alpha^2}\right)\right)\end{array}\right]+(\alpha^2)^{2\mu+n}\Gamma(2\mu+n)\ln\left(\frac{\Omega}{\mu\alpha^2}\right)\right]\right]. \quad (3.10)$$

We could rewrite $\gamma(\mu,x)$ by utilizing the lower normalized incomplete gamma function $P(\mu,x)$ as shown below:

$$\gamma(\mu,x)=\Gamma(\mu)P(\mu,x).$$

Thus, we could re-write $\boldsymbol{H}_3(\mu,\Omega)$ as

$$\boldsymbol{H}_3(\mu,\Omega)=\int_0^{\infty} y^{\mu-1}exp(-\mu y/\Omega)\gamma\left(\mu,\mu\alpha^2y/\Omega\right)\ln\gamma\left(\mu,\mu\alpha^2y/\Omega\right)dy$$

$$=\int_0^{\infty} y^{\mu-1}exp(-\mu y/\Omega)\gamma(\mu,\mu\alpha^2y/\Omega)\,[lnP(\mu,\mu\alpha^2y/\Omega)+\ln\Gamma(\mu)]\;dy\,,$$

as $\partial P(a,x)/\partial x=exp(-x)x^{a-1}/\Gamma(a)$, then through the transformation technique and after some algebra, we could write the following

$$\boldsymbol{H}_3(\mu,\Omega)$$

$$=\frac{\Gamma^2(\mu)}{(\mu\alpha^2/\Omega)^{\mu}}(\mu\alpha^2/\Omega)^{-\mu+1}\int_0^{\infty} u\;exp\left(\left(\vartheta+\sum_{i=1}^{4}\varepsilon_i\vartheta^{i+1}\right)\Big/\alpha^2+2u+\vartheta+\sum_{i=1}^{4}\varepsilon_i\vartheta^{i+1}\right)du$$

$$+\ln\Gamma(\mu)\int_0^{\infty}\mathrm{y}^{\mu-1}exp(-\mu y/\Omega)\gamma(\mu,\mu\alpha^2y/\Omega)dy\,,$$

Here is a number of some notions:

- $\vartheta=\left(exp(\mu)\Gamma(1+\mu)\right)^{1/\mu}$;
- $\varepsilon_1=\frac{1}{\mu+1}$;
- $\varepsilon_2=\frac{3\mu+5}{2(\mu+1)^2(\mu+1)}$;
- $\varepsilon_3=\frac{8\mu^2+33\mu+5}{2(\mu+1)^3(\mu+2)(\mu+3)}$;
- $\varepsilon_4=\frac{125\mu^4+1179\mu^3+3971\mu^2+5661\mu+2888}{24(\mu+1)^4(\mu+2)^2(\mu+3)(\mu+4)}$.

Then

$$\boldsymbol{H}_3(\mu,\Omega)=\frac{\Gamma^2(\mu)}{(\mu\alpha^2/\Omega)^\mu}\int_0^\infty u\,exp\left(\left(1-\frac{1}{\alpha^2}\right)\sum_{i=1}^{4}\varepsilon_i\,\Gamma^{\frac{i+1}{\mu}}(1+\mu)exp\left(\frac{i+1}{\mu}u\right)+2u+\left(1-\frac{1}{\alpha^2}\right)\Gamma^{1/\mu}(1+\mu)exp\left(\frac{u}{\mu}\right)\right)du+4\ln(\mu)+\boldsymbol{H}_1(\mu,\Omega)$$

Let $\alpha_1=\left(1-\frac{1}{\alpha^2}\right)$ and $\alpha_2=\ \Gamma^{1/\mu}(1+\mu)$, then

$$\boldsymbol{H}_3(\mu,\Omega)=\frac{\Gamma^2(\mu)}{(\mu\alpha^2/\Omega)^\mu}\int_0^\infty u\,exp\left(\alpha_1\sum_{i=1}^{4}\varepsilon_i\,\Gamma^{\frac{i+1}{\mu}}(1+\mu)exp\left(\frac{i+1}{\mu}u\right)+2u+\alpha_2 exp\left(\frac{u}{\mu}\right)\right)du+4\ln(\mu)+\boldsymbol{H}_1(\mu,\Omega),$$

After some algebra, we get the following

$$\boldsymbol{H}_3(\mu,\Omega)=\frac{\Gamma^2(\mu)}{(\mu\alpha^2/\Omega)^\mu}\boldsymbol{M}(\alpha,\mu)+4\ln(\mu)\boldsymbol{H}_1(\mu,\Omega) \tag{3.11}$$

where

$$\boldsymbol{M}(\alpha,\mu)=\int_0^\infty u\,exp\left(\alpha_1\sum_{i=1}^{4}\varepsilon_i\,\Gamma^{\frac{i+1}{\mu}}(1+\mu)exp\left(\frac{i+1}{\mu}u\right)+2u+\alpha_2 exp\left(\frac{u}{\mu}\right)\right)du, \tag{3.12}$$

it could be solved by the method of Newton-Raphson, where $\operatorname{Re}\mu(\alpha^2+1)>0$, $\operatorname{Re}\mu>0$ and $\operatorname{Re}(2\mu+1)>0$.

Through substituting (3.9-3.12) in (3.8) we gain the following:

$$H_X\left(f_N^w(x|\mu,\Omega;\alpha)\right)=-\ln\frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu}}{(\Omega\alpha^2)^\mu\,\Gamma(2\mu)\quad{}_2F_1\left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}-\frac{2(\mu)^{1-\mu}(\mu\alpha^2+\mu)^{2\mu}}{(\Omega\alpha^2)^\mu\,\Gamma(2\mu)\ {}_2F_1\ \left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}\times[(2\mu-1)\boldsymbol{H}_1(\mu,\Omega)-(\mu/\Omega)\boldsymbol{H}_2(\mu,\Omega)+\boldsymbol{H}_3(\mu,\Omega)].$$

Further β −entropy can be obtained as follows for $f_N^w(x|\mu,\Omega;\alpha)$:

$$H_\beta\left(f_N^w(x|\mu,\Omega;\alpha)\right)=\frac{1}{\beta-1}\times\left[1-\frac{2(\mu)^{\beta-\beta\mu}(\mu\alpha^2+\mu)^{2\mu\beta}}{(\Omega\alpha^2)^{\mu\beta}\Gamma^\beta(2\mu)\ {}_2F_1^\beta\left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)}\times\int_0^\infty\left(x^{2\beta\mu-\beta}\,exp(-\beta\mu x^2/\Omega)\,\gamma^\beta(\mu,\mu(\alpha x)^2/\Omega)\right)dx\right]. \tag{3.13}$$

Through the alternation of variables to a finite interval and through the usage of Bool's rule (Mathews and Fink (2004)), we could reveal that:

$$A(\alpha,\beta,\mu,\Omega) = \int_0^\infty \left(x^{2\beta\mu-\beta}\, exp(-\,\beta\mu x^2/\Omega)\, \gamma^\beta(\mu, \mu(\alpha x)^2/\Omega)\right) dx, \qquad (3.14)$$

$$\approx \quad \frac{256}{405}(3)^{2\beta\mu+\beta} exp\left(-\frac{\beta\mu}{16\Omega}\right)\gamma^\beta\left(\mu, \frac{\mu\alpha^2}{9\Omega}\right) + \frac{24}{45} exp(-\,\beta\mu/\Omega)\gamma^\beta(\mu, \mu\alpha^2/\Omega) + \frac{256(3)^{2\beta\mu-\beta}}{45} exp(-9\,\beta\mu/\Omega)\gamma^\beta(\mu, 9\,\mu\alpha^2/\Omega).$$

By substituting (3.14) in (3.13), we obtain

$$H_\beta\left(f_N^w(x|\mu,\Omega;\alpha)\right) = \frac{1}{\beta-1}\left[1 - \frac{2(\mu)^{\beta-\beta\mu}(\mu\alpha^2+\mu)^{2\mu\beta})}{(\Omega\alpha^2)^{\mu\beta}\Gamma^\beta(2\mu)\ {}_2F_1^\beta\left(1,2\mu;\mu+1;\frac{\mu\alpha^2}{\mu\alpha^2+\mu}\right)} \times A(\alpha,\beta,\mu,\Omega)\right].$$

The relative loss of entropies in using the Azzeline weighted version of Nakagami-μ distribution $\left(f_N^w(x|\mu,\Omega;\alpha)\right)$ instead of the Nakagami-μ distribution (g) are defined as follows:

$$S_H\left(f_N^w(x|\mu,\Omega;\alpha)\right) = \frac{H_X(g) - H_X f_N^w(x|\mu,\Omega;\alpha)}{H_X(g)},$$

and

$$S_{H_\beta}\left(f_N^w(x|\mu,\Omega;\alpha)\right) = \frac{H_\beta(g) - H_\beta\left(f_N^w(x|\mu,\Omega;\alpha)\right)}{H_\beta(g)}.$$

4. Numerical Results and Comparison for Uncertainty Measures

In the current section, we look into the stochastic ordering of cross-entropy and uncertainty measures for Nakagami$-\mu$ distribution and weighted versions of Nakagami-μ. The outcomes revealed in tables 5-10 are showing the different measures of entropies, while tables 11-16 provide the relative loss of entropies, corresponding to the above six measures due to the usage of size-biased distribution and azzeline weighted version, instead of the distribution of Nakagami.

If X, X^{w^θ} and $X_z{}^w$ are nakagami- μ random variable, size-biased nakagami- μ of order θ random variable and azzeline weighted version of nakagami-μ random variable, with density function $g(x|\mu,\Omega)$and $g^{w\theta}(x|\mu,\Omega;\alpha)$ and $f_N^w(x|\mu,\Omega;\alpha)$ respectively.

As $w(x) = x^{\theta}$is non-decreasing for all $x \geq 0$, by using Theorem 3.1 in Oluyede (2007, pp.951), we gain the following

$$H\left(g(x|\mu,\Omega), g^{w\theta}(x|\mu,\Omega;\alpha)\right) = E\left[-\ln\left(g(x|\mu,\Omega)/g^{w\theta}(x|\mu,\Omega)\right)\right]$$
$$= \int_{0}^{\infty} g(x|\mu,\Omega) \ln \frac{1}{g(x|\mu,\Omega)/g^{w\theta}(x|\mu,\Omega)} dx \leq 0,$$

it leads to

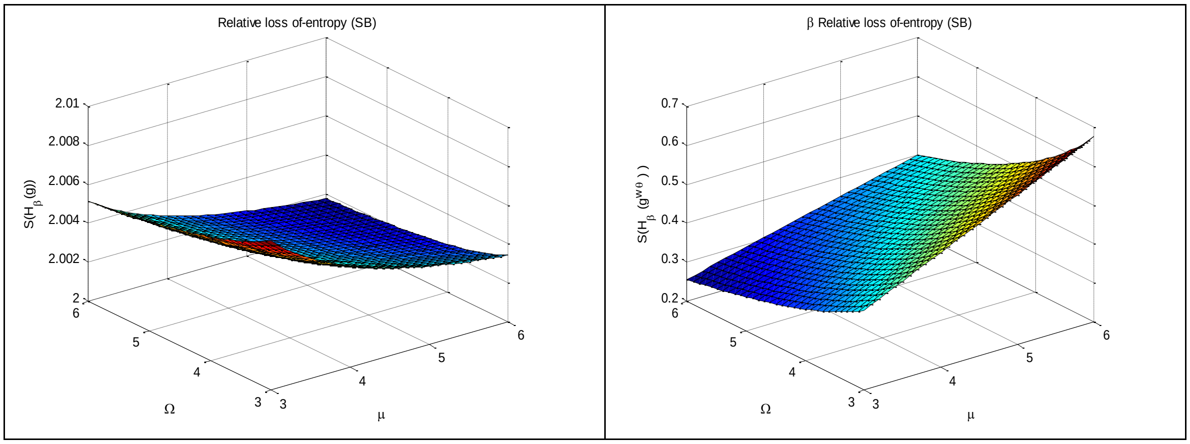
$$H\left(g^{w\theta}(x|\mu,\Omega)\right) \leq H\left(g(x|\mu,\Omega)\right); \text{for } \mu \text{ and } \Omega.$$

This means that the uncertainty present measure of in relation to the value of a random variable X is greater than or equal to the uncertainty present measure in relation to the value of the corresponding weighted random variable $X^{w^{\theta}}$. This note was affirmed by the tables 5 and 7. Thus, we say X is smaller than Y in the differential entropy ordering $\left(X^{w^{\theta}} \leq_D X\right)$. Furthermore, $\Gamma\left(\mu + \frac{\theta}{2}\right)\Gamma(\mu)(\mu/\Omega)^{\frac{\theta}{2}}\theta x^{\theta-1} \geq 0$, then we have $g(x|\mu,\Omega)/g^{w\theta}(x|\mu,\Omega)$ is decreasing in x, as a result, $X \leq_{lr} X^{w^{\theta}}$. From table 6 and 8 we have $H_{\beta}\left(g^{w\theta}(x|\mu,\Omega)\right) > H_{\beta}\left(g(x|\mu,\Omega)\right)$, thus $X \leq_{\beta} X^{w^{\theta}}$. Moreover, from table 8 and 10 we have $H_{\beta}\left(f_N^w(x|\mu,\Omega;\alpha)\right) < H_{\beta}\left(g^{w\theta}(x|\mu,\Omega)\right)$, it leads to ${X_z}^w \leq_{\beta} X^{w^{\theta}}$. Moreover, from table 6 and 10 we get $H_{\beta}\left(f_N^w(x|\mu,\Omega;\alpha)\right) < H_{\beta}\left(g(x|\mu,\Omega)\right)$, then we get ${X_z}^w \leq_{\beta} X$.

As well, through utilizing tables (5-17) we could note that:

- With regard to the fixed μ, and $\beta = 2$ the relative loss of β −entropy is an increasing function in Ω
- With regard to the fixed Ω, the relative loss of entropy is a decreasing function in μ.
- With regard to the fixed Ω and $\beta = 3$ we have the relative loss of entropy as a decreasing function in μ.
- With regard to the $\beta = 2$ and $\beta = 3$, we get have $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega)\right) < S_{H_{\beta}}\left(f_N^w(x|\mu,\Omega;\alpha)\right)$. Moreover, the relative loss of β entropy $S_{H_{\beta}}\left(f_N^w(x|\mu,\Omega;\alpha)\right)$ could be higher than one while $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega)\right)$ is less than one. Therefore the $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega)\right)$is preferable $S_{H_{\beta}}\left(f_N^w(x|\mu,\Omega;\alpha)\right)$.
- With regard to the fixed Ω, we have the relative loss of entropy as an increasing function in μ.
- $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega)\right) < S_H\left(g^{w\theta}(x|\mu,\Omega)\right)$. Moreover, $S_H\left(g^{w\theta}(x|\mu,\Omega)\right)$ could be higher than one while $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega)\right)$ is lower than one. Thus, $S_{H_{\beta}}\left(g^{w\theta}(x|\mu,\Omega;\alpha)\right)$ is preferable $S_H\left(g^{w\theta}(x|\mu,\Omega)\right)$.

Fig 3: Relative loss of Entropies Versions of Nakagami Distributions



Conclusions

In this paper, the differential entropy and β-entropy for Nakagami-μ distributions and their associated distributions were gained. As well, the differential entropy and the β-entropy for the weighted versions of such distributions and their special cases were gained. Moreover, with regard to some certain values of parameters, the differential entropy and the $\beta -$entropy values for Nakagami-μ distributions were derived and summarized in Tables 1 - 4. We revealed that our results reduce to the Shannon entropy as β tends to one.

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Table 5: Entropy for Nakagami distribution $\left(H_X\left(g(x|\mu,\Omega)\right)\right)$

Ω	μ=.1	μ=.25	μ=.35	μ=.5	μ=1	μ=3	μ=3.5	μ=4	μ=4.5	μ=5	μ=5.5	μ=6
1	-1.3586	0.4812	0.6707	0.7258	0.5955	0.1437	0.0720	0.0091	-0.0469	-0.0973	-0.1432	-0.1852
3	-0.8093	1.0305	1.2200	1.2751	1.1448	0.6930	0.6213	0.5584	0.5024	0.4520	0.4061	0.3641
3.5	-0.7323	1.1075	1.2971	1.3522	1.2218	0.7701	0.6984	0.6354	0.5794	0.5290	0.4832	0.4412
4	-0.6655	1.1743	1.3638	1.4189	1.2886	0.8369	0.7651	0.7022	0.6462	0.5958	0.5500	0.5080
4.5	-0.6066	1.2332	1.4227	1.4778	1.3475	0.8958	0.8240	0.7611	0.7051	0.6547	0.6089	0.5669
5	-0.5539	1.2859	1.4754	1.5305	1.4002	0.9485	0.8767	0.8138	0.7578	0.7074	0.6615	0.6195
5.5	-0.5063	1.3335	1.5231	1.5782	1.4478	0.9961	0.9243	0.8614	0.8054	0.7550	0.7092	0.6672
6	-0.4628	1.3770	1.5666	1.6217	1.4913	1.0396	0.9679	0.9049	0.8489	0.7985	0.7527	0.7107

Table 6: $\left(H_\beta\left(g(x|\mu,\Omega)\right)\right)$ with $\beta = 2$

Ω	μ=.1	μ=.5	μ=1	μ=3	μ=3.5	μ=4	μ=4.5	μ=5	μ=5.5	μ=6
.5	1.0263	0.6011	0.5569	0.2917	0.2381	0.1878	0.1404	0.0955	0.0527	0.0117
1	1.0186	0.7179	0.6867	0.4992	0.4612	0.4257	0.3922	0.3604	0.3301	0.3011
3	1.0107	0.8371	0.8191	0.7108	0.6889	0.6684	0.6491	0.6307	0.6133	0.5965
3.5	1.0099	0.8492	0.8325	0.7323	0.7120	0.6930	0.6751	0.6581	0.6419	0.6264
4	1.0093	0.8590	0.8433	0.7496	0.7306	0.7129	0.6961	0.6802	0.6651	0.6506
4.5	1.0088	0.8670	0.8523	0.7639	0.7460	0.7293	0.7135	0.6985	0.6842	0.6706
5	1.0083	0.8738	0.8599	0.7760	0.7591	0.7282	0.7432	0.7140	0.7004	0.6875
5.5	1.0079	0.8797	0.8664	0.7864	0.7703	0.7408	0.7551	0.7273	0.7144	0.7020
6	1.0076	0.8848	0.8721	0.7955	0.7800	0.7655	0.7519	0.7389	0.7265	0.7147

Table 7: Entropy for SB Nakagami distribution $H\left(g^{w\theta}(x|\mu,\Omega)\right)$ for $\theta = 1$

Ω	μ=.1	μ=.25	μ=.35	μ=.5	μ=1	μ=3	μ=3.5	μ=4	μ=4.5	μ=5	μ=5.5	μ=6
1	-1.6104	-1.2247	-1.0877	-0.9420	-0.6496	-0.1491	-0.0758	-0.0120	0.0447	0.0955	0.1417	0.1839
3	-2.1597	-1.7741	-1.6370	-1.4913	-1.1989	-0.6984	-0.6251	-0.5613	-0.5046	-0.4538	-0.4076	-0.3654
3.5	-2.2368	-1.8511	-1.7141	-1.5684	-1.2760	-0.7754	-0.7022	-0.6383	-0.5817	-0.5309	-0.4847	-0.4424
4	-2.3036	-1.9179	-1.7809	-1.6352	-1.3427	-0.8422	-0.7690	-0.7051	-0.6485	-0.5976	-0.5515	-0.5092
4.5	-2.3625	-1.9768	-1.8397	-1.6941	-1.4016	-0.9011	-0.8279	-0.7640	-0.7074	-0.6565	-0.6104	-0.5681
5	-2.4152	-2.0295	-1.8924	-1.7468	-1.4543	-0.9538	-0.8805	-0.8167	-0.7601	-0.7092	-0.6630	-0.6208
5.5	-2.4628	-2.0771	-1.9401	-1.7944	-1.5020	-1.0014	-0.9282	-0.8643	-0.8077	-0.7569	-0.7107	-0.6684
6	-2.5063	-2.1206	-1.9836	-1.8379	-1.5455	-1.0449	-0.9717	-0.9078	-0.8512	-0.8004	-0.7542	-0.7119

Table 8: $H_\beta\left(g^{w\theta}(x|\mu,\Omega)\right)$of order θ=1 and β=2

Ω	μ=.5	μ=1	μ=3	μ=3.5	μ=4	μ=4.5	μ=5	μ=5.5	μ=6
1	0.1138	0.8433	0.9972	0.9988	0.9995	0.9998	0.9999	1.0000	1.0000
3	0.4883	0.9095	0.9984	0.9993	0.9997	0.9999	0.9999	1.0000	1.0000
3.5	0.5263	0.9163	0.9985	0.9994	0.9997	0.9999	0.9999	1.0000	1.0000
4	0.5569	0.9217	0.9986	0.9994	0.9997	0.9999	0.9999	1.0000	1.0000
4.5	0.5822	0.9261	0.9987	0.9994	0.9998	0.9999	1.0000	1.0000	1.0000
5	0.6037	0.9299	0.9988	0.9995	0.9998	0.9999	1.0000	1.0000	1.0000
5.5	0.6221	0.9332	0.9988	0.9995	0.9998	0.9999	1.0000	1.0000	1.0000
6	0.6382	0.9360	0.9989	0.9995	0.9998	0.9999	1.0000	1.0000	1.0000

Table 9: Entropy $H_X\left(f_N^w(x|\mu,\Omega;\alpha)\right)$ with α=0.5 and $\mu = \{0.5, 1, 3, 3.5, 4\}$

Ω	μ=.5	μ=1	μ=3	μ=3.5	μ=4
1	0.4061×10^{2}	-0.5591×10^{4}	-7.5606×10^{15}	-0.9424×10^{16}	-0.9568×10^{18}
3	-0.7123×10^{2}	0.1856×10^{4}	-6.2018×10^{15}	-0.6485×10^{16}	-0.3344×10^{18}
3.5	-0.7701×10^{2}	0.3712×10^{4}	-5.3722×10^{15}	-0.4374×10^{16}	0.1976×10^{18}
4	-0.8238×10^{2}	0.5567×10^{4}	-4.2682×10^{15}	-0.1363×10^{16}	1.0127×10^{18}
4.5	-0.8742×10^{2}	0.7422×10^{4}	-2.8507×10^{15}	0.2749×10^{16}	2.1973×10^{18}
5	-0.9219×10^{2}	0.9275×10^{4}	-1.0805×10^{15}	0.8172×10^{16}	3.8493×10^{18}
5.5	-0.9672×10^{2}	1.1128×10^{4}	1.0818×10^{15}	1.5132×10^{16}	6.0778×10^{18}
6	-1.0104×10^{2}	1.2980×10^{4}	3.6751×10^{15}	2.3864×10^{16}	9.0037×10^{18}

Table 10: Entropy $H_X(f_N^w(x|\mu,\Omega;\alpha))$ with α=0.5 and $\mu = \{4.5, 5, 5.5, 6\}$

Ω	μ=4.5	μ=5	μ=5.5	μ=6
1	-0.0827×10^{21}	-0.0621×10^{25}	-0.0041	-0.0024
3	0.0514×10^{21}	0.2314×10^{25}	0.0611×10^{26}	0.1444×10^{28}
3.5	0.1858×10^{21}	0.5727×10^{25}	0.1482×10^{26}	0.3678×10^{28}
4	0.4071×10^{21}	1.1756×10^{25}	0.3133×10^{26}	0.8222×10^{28}
4.5	0.7493×10^{21}	2.1679×10^{25}	0.6023×10^{26}	1.6687×10^{28}
5	1.2537×10^{21}	3.7134×10^{25}	1.0781×10^{26}	3.1409×10^{28}
5.5	1.9689×10^{21}	6.0168×10^{25}	1.8232×10^{26}	5.5642×10^{28}
6	2.9514×10^{21}	9.3276×10^{25}	2.9439×10^{26}	9.3772×10^{28}

Table 11: $H_\beta(f_N^w(x|\mu,\Omega;\alpha))$ with $\beta = 2$ and $\alpha = 0.5$

Ω	μ=.5	μ=1	μ=3	μ=3.5	μ=4	μ=4.5	μ=5	μ=5.5	μ=6
1	0.1244	-5.1957	-33.0597	-24.5532	-14.6913	-7.1261	-2.6318	-0.4262	0.5009
3	0.5559	-1.2204	-1.7699	-0.3236	0.4804	0.8273	0.9503	0.9874	0.9972
3.5	0.5744	-1.2164	-1.0645	0.1198	0.6929	0.9095	0.9770	0.9948	0.9990
4	0.5843	-1.3829	-2.0340	-0.4154	0.4563	0.8227	0.9498	0.9875	0.9972
4.5	0.5900	-1.6353	-4.7859	-2.1525	-0.4156	0.4610	0.8222	0.9484	0.9866
5	0.5938	-1.9139	-9.1383	-5.1613	-2.0727	-0.2948	0.5287	0.8492	0.9570
5.5	0.5970	-2.1815	-14.5494	-9.1702	-4.4436	-1.4578	0.0424	0.6723	0.9001
6	0.6001	-2.4178	-20.3182	-13.6682	-7.2478	-2.9090	-0.5982	0.4261	0.8164

Table 12: Relative loss of β −entropy ($\beta = 2$)

μ	$\Omega = 3$		$\Omega = 4$		$\Omega = 4.5$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$)	$S_{H_\beta}(f_N^w)$
.5	0.4167	0.3359	0.3517	0.3198	0.3285	0.3195
1	-0.1104	2.4899	-0.0929	2.6398	-0.0867	2.9187
3	-0.4045	3.4899	-0.3322	3.7136	-0.3073	7.2650
3.5	-0.4505	1.4697	-0.3679	1.5686	-0.3397	3.8853
4	-0.4956	0.2813	-0.4025	0.3599	-0.3709	1.5699
4.5	-0.5404	-0.2746	-0.4364	-0.1818	-0.4014	0.3539
5	-0.5853	-0.5067	-0.4701	-0.3963	-0.4316	-0.1771
5.5	-0.6306	-0.6101	-0.5036	-0.4847	-0.4615	-0.3861
6	-0.6764	-0.6716	-0.5371	-0.5328	-0.4913	-0.4714

Table 13: Relative loss of β −entropy ($\beta = 2$)

μ	$\Omega = 5$		$\Omega = 5.5$		$\Omega = 6$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$
.5	0.3092	0.3205	0.2928	0.3214	0.2787	0.3214
1	-0.0815	3.2258	-0.0771	3.5179	-0.0733	3.5179
3	-0.2870	12.7758	-0.2700	19.5003	-0.2556	19.5003
3.5	-0.3167	7.7997	-0.2976	12.9052	-0.2814	12.9052
4	-0.3453	3.7890	-0.3240	6.8847	-0.3060	6.8847
4.5	-0.3731	1.4049	-0.3497	2.9678	-0.3299	2.9678
5	-0.4005	0.2595	-0.3749	0.9417	-0.3533	0.9417
5.5	-0.4277	-0.2124	-0.3998	0.0589	-0.3764	0.0589
6	-0.4546	-0.3921	-0.4245	-0.2822	-0.3992	-0.2822

Table 14: Relative loss of β −entropy ($\beta = 3$)

μ	$\Omega = 3$		$\Omega = 4$		$\Omega = 4.5$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$
.5	0.0066	0.1334	$S_{SB}(\beta)$	$S_{AZ}(\beta)$	0.0044	0.0884
1	-0.0351	2.0222	0.0049	0.0980	-0.0231	1.4194
3	-0.1062	4.5587	-0.0261	1.4248	-0.0684	4.1475
3.5	-0.1251	1.4743	-0.0776	1.7248	-0.0801	1.6435
4	-0.1448	0.2608	-0.0910	0.4670	-0.0921	0.4367
4.5	-0.1651	-0.0853	-0.1048	0.0258	-0.1043	0.0223
5	-0.1862	-0.1736	-0.1189	-0.0948	-0.1169	-0.0925
5.5	-0.2081	-0.2064	-0.1334	-0.1298	-0.1297	-0.1258
6	-0.2308	-0.2306	-0.1483	-0.1479	-0.1429	-0.1424

Table 15: Relative loss of β −entropy ($\beta = 3$)

μ	$\Omega = 5$		$\Omega = 5.5$		$\Omega = 6$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$
.5	0.0039	0.0816	0.0036	0.0765	0.0033	0.0726
1	-0.0207	1.5101	-0.0188	1.6466	-0.0172	1.7950
3	-0.0611	9.8300	-0.0553	18.8792	-0.0504	30.3960
3.5	-0.0715	4.7071	-0.0646	10.1035	-0.0589	17.5239
4	-0.0821	1.6289	-0.0741	3.9568	-0.0675	7.4190
4.5	-0.0929	0.3818	-0.0838	1.1522	-0.0763	2.3884
5	-0.1040	0.0016	-0.0936	0.2100	-0.0852	0.5647
5.5	-0.1153	-0.0959	-0.1037	-0.0423	-0.0942	0.0468
6	-0.1268	-0.1238	-0.1139	-0.1035	-0.1035	-0.0777

Table 16: Relative loss of entropy

μ	$\Omega = 3$		$\Omega = 4$		$\Omega = 4.5$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$
.5	2.1696	-54.8631	2.1524	-57.0571	2.1463	-58.1549
1	2.0473	-0.1620×10^{4}	2.0420	-0.4319×10^{4}	2.0402	-0.5507×10^{4}
3	2.0077	0.8949×10^{14}	2.0064	0.5100×10^{14}	2.0059	0.3182×10^{14}
3.5	2.0062	0.1044×10^{17}	2.0050	0.0178×10^{17}	2.0047	-0.0334×10^{17}
4	2.0052	0.0060×10^{20}	2.0041	-0.0144×10^{20}	2.0038	-0.0289×10^{20}
4.5	2.0045	$-0.1022 10^{21}$	2.0035	-0.6300×10^{21}	2.0032	-1.0627×10^{21}
5	2.0040	-0.0512×10^{24}	2.0031	-0.1973×10^{24}	2.0028	-0.3311×10^{24}
5.5	2.0037	-0.1504×10^{26}	2.0027	-0.5696×10^{26}	2.0025	-0.9892×10^{26}
6	2.0034	-0.0397×10^{29}	2.0025	-0.1619×10^{29}	2.0022	-0.2944×10^{29}

Table 17: Relative loss of entropy

μ	$\Omega = 5$		$\Omega = 5.5$		$\Omega = 6$	
	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$S_{H_\beta}(g^{w\theta})$	$S_{H_\beta}(f_N^w)$	$\boldsymbol{S_{H_\beta}(g^{w\theta})}$)	$S_{H_\beta}(f_N^w)$
.5	2.1413	-59.2323	2.1370	-60.2838	2.1333	-61.3074
1	2.0387	-0.6623×10^{4}	2.0374	-0.7685×10^{4}	2.0363	-0.8703×10^{4}
3	2.0056	0.1139×10^{14}	2.0053	-0.1086×10^{14}	2.0051	-0.3535×10^{14}
3.5	2.0044	-0.0932×10^{17}	2.0042	-0.1637×10^{17}	2.0040	-0.2466×10^{17}
4	2.0036	-0.0473×10^{20}	2.0034	-0.0706×10^{20}	2.0032	-0.0995×10^{20}
4.5	2.0030	-1.6544×10^{21}	2.0028	-2.4444×10^{21}	2.0027	-3.4765×10^{21}
5	2.0026	-0.5250×10^{24}	2.0024	-0.7969×10^{24}	2.0023	-1.1681×10^{24}
5.5	2.0023	-1.6296×10^{26}	2.0021	-2.5708×10^{26}	2.0020	-3.9111×10^{26}
6	2.0020	-0.5070×10^{29}	2.0019	-0.8340×10^{29}	2.0018	-1.3194×10^{29}