

Statistical Analysis of the TM–Model via Bayesian Approach

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Abstract

The method of paired comparisons calls for the comparison of treatments presented in pairs to judges who prefer to better one based on their sensory evaluations. Thurstone (1927) and Mosteller (1951) employ the method of maximum likelihood to estimate the parameters of the Thurstone-Mosteller model for the paired comparisons. A Bayesian analysis of the said model using the non-informative reference (Jeffreys) prior is presented in this study. The posterior estimates (means and joint modes) of the parameters and the posterior probabilities comparing the two parameters are obtained for the analysis. The predictive probabilities that one treatment (T_i) is preferred to any other treatment (T_j) in a future single comparison are also computed. In addition, the graphs of the marginal posterior distributions of the individual parameter are drawn. The appropriateness of the model is also tested using the different test-statistics.

Keywords: Paired comparison method, Thurstone-Mosteller model, Posterior distribution, Non-informative prior, Jeffreys prior, Predictive distribution, Bayesian hypotheses testing.

1. Introduction

In the method of paired comparisons (PC), judges are presented with pairs of treatments and for each pair, they are asked to choose the best one according to some criterion. The method of PC has been widely employed to remove the difficulties involved in simultaneously comparison of several treatments/objects. This method implies the pairwise comparison of the objects and ranks the objects by means of a score. Scientists and practitioners from various fields and backgrounds are practicing the PC technique in various tasks and researches. For the analysis of PC data, numerous models have been

developed. Fluctuations in the evaluations of merits of competing treatments are captured by a random variable, which is assumed to be identically and independently distributed for all the pairs of treatments. Different distributional assumptions of the random variable lead to different models (David, 1988). The literature reveals different situations in which the method of PC is used and it also discusses various models devoted to study these situations. For instance, Bradley (1976), David (1988) and Davidson and Farquhar (1976) provide a detailed review of the PC models. Bradley (1953) assumes the responses following Logistic distribution and proposes the model. McCullagh (2000) advocates his model as a special case of a logistic analysis of variance model for binomial data. Stern (1990) considers an approach to build models for PC on comparing two gamma random variables and establishes the Bradley-Terry (BT) model as a special case of the gamma models. Abbas and Aslam (2009) consider Cauchy distribution to build PC model. Rao and Kupper (RK)(1967) and Davidson (1970) extend the basic models by including the effects of ties. Davidson and Beaver (1977) extend the BT model to accommodate the within-pair order effects. Aslam (2001, 2002) presents the Bayesian analyses of the paired comparisons models (BT and RK) using reference prior.

The Thurstone-Mosteller model is defined and discussed in section 2. The notations for the model are given in section 3 with the likelihood function of the model. Section 4 consists of the Bayesian analysis of the model using the reference (Jeffreys) prior for three treatment parameters. The Jeffreys prior is defined and derived in this section. The posterior estimates (means and joint modes) of the parameters are determined. The posterior probabilities are calculated for the Bayesian testing of hypotheses and the predictive probabilities are also calculated for a single future comparison. Section 5 covers appropriateness of the model for three treatments. Last section 6 presents the conclusion of the analysis of the said model.

2. The Thurstone-Mosteller Model

In the paired comparison experiments, let we have m treatments T_1, T_2, \dots, T_m , which are to be compared in pairs. Each pair (T_i, T_j) , $1 \leq i < j \leq m$ is ranked r_{ij} times. If there are n judges then total number of paired comparisons will be $n {}^m C_2$.

The Bradley-Terry (1952) developed the PC model which implies that the difference between two latent variables $(X_i - X_j)$ has a logistic density with parameter $(\ln \theta_i - \ln \theta_j)$. If ψ_{ij} denotes the probability $P(X_i > X_j | \theta_i, \theta_j)$ that the treatment T_i is preferred to the treatment T_j ($i \neq j$) when treatment T_i and treatment T_j are compared then it is defined as:

$$\begin{aligned} \psi_{ij} &= \frac{1}{4} \int_{-(\ln \theta_i - \ln \theta_j)}^{\infty} \sec^2(y/2) dy = \int_{-(\ln \theta_i - \ln \theta_j)}^{\infty} \frac{e^{-y}}{(1 + e^{-y})} dy \\ &= \frac{\theta_i}{\theta_i + \theta_j} \end{aligned}$$

Thurstone (1927) used the normal distribution in applications of the model. To compare m treatments with the restriction that no tie will occur, specifically we assume that for treatments T_i and T_j with respective parameters θ_i and θ_j if $X_j > X_i$ where X_i and X_j are response (latent) variables and according to the Thurstone-Mosteller model, the probability distribution of the difference $(X_i - X_j)$ is normal with mean $(\theta_i - \theta_j)$ and unit variance. The probability that treatment T_i is preferred to treatment T_j is denoted by ψ_{ij} , the probability that treatment T_j is preferred to treatment T_i is denoted by ψ_{ji} , the model may be summarized as:

$$\psi_{ij} = P(X_i > X_j) = \frac{1}{\sqrt{2\pi}} \int_{-(\theta_i - \theta_j)}^{\infty} e^{-\frac{1}{2}y^2} dy,$$

$$\psi_{ij} = P(X_i > X_j) = \Phi(\theta_i - \theta_j) \quad \text{and} \quad \psi_{ji} = P(X_j > X_i) = \Phi(\theta_j - \theta_i), \quad (1)$$

where Φ is the standard normal cumulative distribution function.

3. Notations and Likelihood Function of the Model

The following notations are used in the analysis of the Thurstone-Mosteller model.

$n_{ijk} = 1$ or 0 according as treatment T_i is preferred to treatment T_j or not in the k th repetition ($k=1,2,3,\dots,r$) of the comparisons.

$r_{ij} =$ Total number of times that treatment T_i is compared with treatment T_j .

$n_{ij} =$ The number of times that treatment T_i is preferred to treatment T_j and $n_{ij} = \sum_k n_{ijk}$

$n_{ji} =$ The number of times that treatment T_j is preferred to treatment T_i and $n_{ji} = \sum_k n_{jik}$

It is to be noted that $n_{ijk} + n_{jik} = 1$ and $n_{ij} + n_{ji} = r_{ij}$.

The probability of the observed result in the k th repetition of the treatments pair (T_i, T_j) according to the Thurstone-Mosteller model is

$$P_{ijk} = \{\Phi(\theta_i - \theta_j)\}^{n_{ijk}} \{1 - \Phi(\theta_i - \theta_j)\}^{n_{jik}} = \{\Phi(\theta_i - \theta_j)\}^{n_{ijk}} \{\Phi(\theta_j - \theta_i)\}^{n_{jik}}$$

Hence the likelihood function of the observed outcome {represents the data (r_{ij}, n_{ij}, n_{ji}) } is

$$L(\mathbf{x}; \theta_1, \theta_2, \dots, \theta_m) = \prod_{i < j=1}^m \prod_{k=1}^{r_{ij}} P_{ijk}$$

$$L(\mathbf{x}; \theta_1, \theta_2, \dots, \theta_m) = \prod_{i < j=1}^m \binom{r_{ij}}{n_{ij}} \prod_{i \neq j=1}^m \{\Phi(\theta_i - \theta_j)\}^{n_{ij}} \quad (2)$$

where $-\infty \leq \theta_i \leq +\infty$, ($i=1,2,3,\dots,m$), these $\theta_1, \theta_2, \dots, \theta_m$ are the worth parameters with a restriction that their sum is zero which ensures that the parameters are well defined.

4. Bayesian Analysis of the Model for $M=3$

Let us consider the case for the comparison of three treatments $m=3$. There are three parameters $(\theta_1, \theta_2, \theta_3)$ when three treatments are compared pair-wise.

The likelihood function of the model is

$$L(x, \theta_1, \theta_2, \theta_3) = C \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(\theta_1 - \theta_3) \}^{n_{13}} \\ \{ \Phi(\theta_3 - \theta_1) \}^{n_{31}} \{ \Phi(\theta_2 - \theta_3) \}^{n_{23}} \{ \Phi(\theta_3 - \theta_2) \}^{n_{32}}$$

where $C = \binom{r_{12}}{n_{12}} \binom{r_{13}}{n_{13}} \binom{r_{23}}{n_{23}}$, using the constraint: $\theta_1 + \theta_2 + \theta_3 = 0$ then $\theta_3 = -(\theta_1 + \theta_2)$,

Now the likelihood function is

$$L(x, \theta_1, \theta_2) = C \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(2\theta_1 + \theta_2) \}^{n_{13}} \\ \{ \Phi(-2\theta_1 - \theta_2) \}^{n_{31}} \{ \Phi(\theta_1 + 2\theta_2) \}^{n_{23}} \{ \Phi(-\theta_1 - 2\theta_2) \}^{n_{32}}$$

$$\ln L(x; \theta_1, \theta_2) = \ln C + n_{12} \ln \Phi(\theta_1 - \theta_2) + n_{21} \ln \Phi(\theta_2 - \theta_1) + n_{13} \ln \Phi(2\theta_1 + \theta_2) + \\ n_{31} \ln \Phi(-2\theta_1 - \theta_2) + n_{23} \ln \Phi(\theta_1 + 2\theta_2) + n_{32} \ln \Phi(-\theta_1 - 2\theta_2) \quad (3)$$

4.1 Jeffreys Prior for the Parameters of the Model

A non-informative prior has been suggested by Jeffreys (1946, 1961) which is frequently used in the situation where one does not have much information about the parameters. It is defined as the density of the parameters proportional to the square root of the determinant of the Fisher's Information matrix. If θ is a $(k \times 1)$ parameters vector then the Fisher's information is

$$I(\theta) = -E \left\{ \frac{\partial^2 \ln L(\mathbf{x}; \theta)}{\partial \theta_i \partial \theta_j} \right\}$$

The Jeffreys prior is derived as $p(\theta) \propto \sqrt{\det\{I(\theta)\}}$.

Derivation of the Jeffreys Prior

Let $\eta = (\theta_1, \theta_2)$, the determinant of Fisher's information matrix is:

$$\det\{I(\eta)\} = (-1)^2 \begin{vmatrix} \frac{E\{\partial^2 l(\eta)\}}{\partial \theta_1^2} & \frac{E\{\partial^2 l(\eta)\}}{\partial \theta_1 \partial \theta_2} \\ \frac{E\{\partial^2 l(\eta)\}}{\partial \theta_2 \partial \theta_1} & \frac{E\{\partial^2 l(\eta)\}}{\partial \theta_2^2} \end{vmatrix}$$

where $l(\eta) = \ln L(\mathbf{x}; \theta_1, \theta_2)$ and

$$E \left(\frac{\partial^2 l(\eta)}{\partial \theta_1^2} \right) = E \left[\frac{n_{12} \{-(\theta_1 - \theta_2) \phi(\theta_1 - \theta_2) \Phi(\theta_1 - \theta_2) - \phi(\theta_1 - \theta_2)^2\}}{\Phi(\theta_1 - \theta_2)^2} \right] - \\ E \left[\frac{n_{21} \{(\theta_2 - \theta_1) \phi(\theta_2 - \theta_1) \Phi(\theta_2 - \theta_1) + \phi(\theta_2 - \theta_1)^2\}}{\Phi(\theta_2 - \theta_1)^2} \right] + 4 \\ E \left[\frac{n_{13} \{-(2\theta_1 + \theta_2) \phi(2\theta_1 + \theta_2) \Phi(2\theta_1 + \theta_2) - \phi(2\theta_1 + \theta_2)^2\}}{\Phi(2\theta_1 + \theta_2)^2} \right] - 4$$

$$\begin{aligned}
 & E \left[\frac{n_{31} \{(-2\theta_1 - \theta_2)\phi(-2\theta_1 - \theta_2)\Phi(-2\theta_1 - \theta_2) + \phi(-2\theta_1 - \theta_2)^2\}}{\Phi(-2\theta_1 - \theta_2)^2} \right] + \\
 & E \left[\frac{n_{23} \{-(\theta_1 + 2\theta_2)\phi(\theta_1 + 2\theta_2)\Phi(\theta_1 + 2\theta_2) - \phi(\theta_1 + 2\theta_2)^2\}}{\Phi(\theta_1 + 2\theta_2)^2} \right] + \\
 & E \left[\frac{n_{32} \{(-\theta_1 - 2\theta_2)\phi(-\theta_1 - 2\theta_2)\Phi(-\theta_1 - 2\theta_2) + \phi(-\theta_1 - 2\theta_2)^2\}}{\Phi(-\theta_1 - 2\theta_2)^2} \right] \quad (4)
 \end{aligned}$$

Similarly the other elements of the determinant are obtained. It is difficult to simplify the determinant, so it can be used numerically.

4.2 The Posterior Distribution of the Parameters

The joint posterior distribution for parameters θ_1, θ_2 and θ_3 given data \mathbf{x} is:

$$\begin{aligned}
 p(\theta_1, \theta_2 | \mathbf{x}) & \propto p_J(\theta_1, \theta_2, \theta_3) L(\mathbf{x}, \theta_1, \theta_2, \theta_3) \\
 p(\theta_1, \theta_2 | \mathbf{x}) & = k_1^{-1} p_J(\theta_1, \theta_2, \theta_3) \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(2\theta_1 + \theta_2) \}^{n_{13}} \\
 & \quad \{ \Phi(-2\theta_1 - \theta_2) \}^{n_{31}} \{ \Phi(\theta_1 + 2\theta_2) \}^{n_{23}} \{ \Phi(-\theta_1 - 2\theta_2) \}^{n_{32}}, \quad (5)
 \end{aligned}$$

where $p_J(\theta_1, \theta_2, \theta_3)$ is the Jeffreys prior distribution, k_1^{-1} is normalizing constant, $-\infty \leq \theta_1, \theta_2 \leq \infty$ and $\theta_1 + \theta_2 + \theta_3 = 0$, so $\theta_3 = -(\theta_1 + \theta_2)$.

The marginal posterior distribution of θ_1 is:

$$\begin{aligned}
 p(\theta_1 | \mathbf{x}) & = \int_{-\infty}^{\infty} k_1^{-1} p_J(\theta_1, \theta_2, \theta_3) \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(2\theta_1 + \theta_2) \}^{n_{13}} \\
 & \quad \{ \Phi(-2\theta_1 - \theta_2) \}^{n_{31}} \{ \Phi(\theta_1 + 2\theta_2) \}^{n_{23}} \{ \Phi(-\theta_1 - 2\theta_2) \}^{n_{32}} d\theta_2, \quad (6a) \\
 & \quad -\infty \leq \theta_1 \leq \infty
 \end{aligned}$$

$$\begin{aligned}
 p(\theta_2 | \mathbf{x}) & = \int_{-\infty}^{\infty} k_1^{-1} p_J(\theta_1, \theta_2, \theta_3) \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(2\theta_1 + \theta_2) \}^{n_{13}} \\
 & \quad \{ \Phi(-2\theta_1 - \theta_2) \}^{n_{31}} \{ \Phi(\theta_1 + 2\theta_2) \}^{n_{23}} \{ \Phi(-\theta_1 - 2\theta_2) \}^{n_{32}} d\theta_1, \quad (6b) \\
 & \quad -\infty \leq \theta_2 \leq \infty
 \end{aligned}$$

and

$$\begin{aligned}
 p(\theta_3 | \mathbf{x}) & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_1^{-1} p_J(\theta_1, \theta_2, \theta_3) \{ \Phi(\theta_1 - \theta_2) \}^{n_{12}} \{ \Phi(\theta_2 - \theta_1) \}^{n_{21}} \{ \Phi(2\theta_1 + \theta_2) \}^{n_{13}} \\
 & \quad \{ \Phi(-2\theta_1 - \theta_2) \}^{n_{31}} \{ \Phi(\theta_1 + 2\theta_2) \}^{n_{23}} \{ \Phi(-\theta_1 - 2\theta_2) \}^{n_{32}} d\theta_1 d\theta_2, \quad (6c) \\
 & \quad -\infty \leq \theta_3 \leq \infty
 \end{aligned}$$

The following simulated data set is used for drawing graphs and further analysis.

Table 1: Simulated Data Set for m=3

Pairs (i,j)	n_{ij}	n_{ji}	r_{ij}
(1,2)	18	12	30
(1,3)	14	16	30
(2,3)	7	23	30

The graphical presentation of the marginal posterior distributions is given below.

Posterior (Marginal) Densities of the Parameters

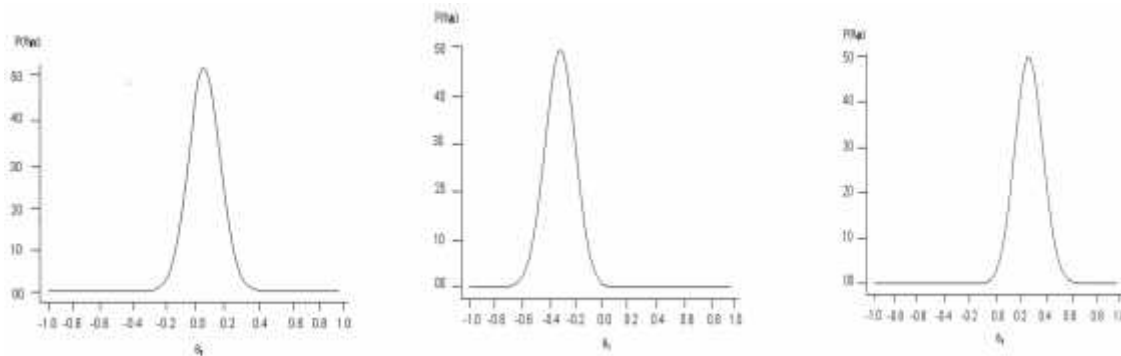


Figure 1

4.3 Posterior Estimates

The posterior means and the joint modes of the parameters using the Jeffreys prior are considered as the posterior estimates of the parameters.

- Posterior Means: The posterior means for the parameters θ_1, θ_2 and θ_3 using the Quadrature method are obtained to be 0.05338, -0.31925 and 0.26192.
- Joint Posterior Modes: As the Jeffreys prior is not in closed form, so it is difficult to find the joint posterior modes mathematically so the posterior modes are obtained by sorting the maximum value of the densities. We observe that the densities are maximum at 0.0500, -0.3250 and 0.2750 which are very close to the calculated values of the posterior means and hence the similar ranking of the treatments is observed.

4.4 Posterior Probabilities of the Hypotheses

Let us consider the hypotheses

$$H_{12}: \theta_1 \geq \theta_2 \text{ vs } \bar{H}_{12}: \theta_1 < \theta_2$$

The posterior probability p_{12} of H_{12} is evaluated from the expression:

$$p_{12} = p(\theta_1 > \theta_2) = \int_{\phi=0}^{\infty} \int_{\theta_3=-\infty}^{\infty} \int_{\xi=-\infty}^{\infty} p(\phi, \xi, \theta_3 | x) d\xi d\theta_3 d\phi, \quad (8)$$

where $\phi = \theta_1 - \theta_2$ and $\xi = \theta_1$. It is to be noted that $q_{12} = P(\theta_1 < \theta_2) = 1 - p_{12}$.

The posterior probabilities are obtained by using quadrature method, we get the posterior probabilities as $p_{12}=0.98312$ and $q_{12}=0.01688$. We accept the hypothesis H_{12} that the parameter θ_1 is greater than the parameter θ_2 . Similarly we obtain $p_{13}=0.17064$ with $q_{13}=0.82936$, the decision is inconclusive for H_{13} and \bar{H}_{13} but $p_{23}=0.0018769$ and $q_{23}=0.998123$ where H_{23} is rejected and \bar{H}_{23} is accepted with high probability.

4.5 Predictive Probabilities

The chance that treatment T_1 is preferred to T_2 in a single future comparison under the TM model is computed in terms of predictive probability $P_{(12)}$ as

$$P_{(12)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\theta_1 - \theta_2) p(\theta_1, \theta_2) d\theta_1 d\theta_2, \quad (9)$$

where $\Phi(\theta_1 - \theta_2)$ is the preference probability of T_1 over T_2 as defined in the model and $p(\theta_1, \theta_2 | \mathbf{x})$ is the joint posterior distribution.

The predictive probabilities are obtained using quadrature method in SAS package. The value of the predictive probability $P_{(12)}$ is computed to be 0.64618. Similarly we get the other predictive probabilities as $P_{(13)} = 0.0048827$ and $P_{(23)} = 0.28155$.

5. Appropriateness of the Tm Model

To test the hypotheses that the fit is good or appropriateness of the TM model for paired comparisons, we have the following hypotheses:

H_0 : The model is appropriate for any value of the parameter $\theta = \theta_0$

H_1 : The model is not appropriate for any value of the parameter θ

where θ is the vector of parameters.

For testing the appropriateness of the model, we compared the observed and expected number of preferences. The expected number of preferences are calculated using the following expressions:

$$\hat{n}_{ij} = r_{ij} \{ \Phi(\theta_i - \theta_j) \} \quad \text{and} \quad \hat{n}_{ji} = r_{ij} \{ \Phi(\theta_j - \theta_i) \} \quad (10)$$

For this purpose, we use the usual χ^2 test with $(m-1)(m-2)/2$ degrees of freedom.

The observed and expected number of preferences are given in Table 2.

Table 2: The Observed and Expected Number of Preferences

Pairs(i,j)	n_{12}	n_{21}	n_{13}	n_{31}	n_{23}	n_{32}
Obs. Freq.	18	12	14	16	07	23
Exp. Freq.	19	11	12	18	08	22

The values of the different test-statistics are given below in Table 3.

Table 3: Goodness of Fit Tests

Test Name	Test Statistics	P-Value
Pearson χ^2	0.867000	0.352426
Anderson-Darling	0.323304	0.541112
Kolmogorov-Smirnov	0.227092	0.529792
Kuiper	0.454183	0.267540
Shapiro-Wilk	0.937117	0.636527

Hence from the above table, it is clear that there is no evidence that the model does not give good fit.

Comments and Conclusion

The Thurstone-Mosteller model is selected for a Bayesian analysis using the reference (Jeffreys) prior. The reference (Jeffreys) prior is derived for three treatment parameters. The prior has no closed form so it is used numerically by designing a program in SAS package. The posterior estimates (means and joint modes) of the parameters are determined. The posterior probabilities are evaluated for the Bayesian testing of hypotheses for comparing two parameters. The predictive probabilities are calculated for a single future comparisons of two treatments. The appropriateness of the model is also tested through different tests which indicate that the fit is good.

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