On Generalized Log Burr Xii Distribution

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Abstract

In this paper, we present flexible generalized log Burr XII (GLBXII) distribution developed on the basis of generalized log Pearson differential equation. GLBXII distribution is also obtained from compounding mixture of distributions. Some structural and mathematical properties including moments, inequality measures, uncertainty measures and reliability measures are theoretically established. Characterizations of GLBXII distribution are also studied through different techniques. Parameters of GLBXII distribution are estimated using maximum likelihood method. Goodness of fit of probability distribution through different methods is studied.

Keywords: Moments; Uncertainty; Reliability; Characterizations; Maximum Likelihood

1. Introduction

Burr (1942) suggested 12 distributions as Burr family to fit cumulative frequency functions on frequency data. The Burr-XII (BXII) distribution is widely used in modeling lifetime, finance and insurance data analysis. The Burr-XII (BXII) distribution also has wide applications in reliability and acceptance sampling plans. In many areas of applications, the Burr distribution of any type is inadequate and as such various generalizations and extensions of the Burr distribution have been proposed in the statistical literature.

The cumulative distribution function (cdf) for random variable Y having Burr XII distribution is

$$G(y)=1-\left(1+y^{\beta}\right)^{-\alpha} \quad y\geq 0,\alpha>0,\beta>0. \tag{1}$$

The probability density function (pdf) of Burr XII distribution for random variable Y is

$$g(y)=\alpha\beta y^{(\beta-1)}\left(1+y^{\beta}\right)^{-(\alpha+1)}, \mathrm{y}>0,\alpha>0,\beta>0. \tag{2}$$

Recently, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional threshold/shape/scale parameter(s) or using some transformation to the baseline distribution.

Many authors like Zimmer and Burr (1963), Takahasi (1965), Tadikamalla (1980), Saran and Pushkarna (1999), Begum and Parvin (2002), Shao et al. (2004), Olapade (2008), Paranaiba et al. (2011), Usta (2013) and Paranaiba et al. (2013) studied BXII distribution. Okasha et al. (2015) studied BXII distribution and its various properties. Silver and Cordeiro (2015) developed a new compounding family by mixing BXII and power series distribution. Gomes et al. (2015) presented McDonald Burr XII distribution along with properties and application.

Muhammad (2016) established generalized BXII- Poisson distribution to analyze strength of material. Afify et al. (2016) considered Weibull BXII distribution for analyzing the strengths of glass fibers. Thupeng (2016) modeled concentrations of daily extreme nitrogen dioxide with BXII distribution. Doğru and Arslan (2016) estimated the parameters of BXII distribution with optimal B-robust estimators. Yari and Tondpour (2017) derived a new Burr distribution to study the lifetime cancer data. Ghosh and Bourguignon (2017) presented properties of the extended BXII distribution. Mdlongwa et al. (2017) studied properties and applications of BXII modified Weibull distribution. Mead and Afify (2017) also presented properties and applications of BXII distribution with five parameters. Bhatti et al. (2017) have studied the generalized log Pearson differential equation (GLPE) to develop probability distributions with various choices of the coefficients. Guerra et al. (2017) developed gamma burr XII distribution with flexible hazard function and studied different properties and application. Cadena (2017) studied different behaviors of extended Burr XII distribution. Kumar (2017) studied different properties of Burr type XII distribution.

In this paper, a new family of distribution is studied, that is more flexible probability model in fitting lifetime data, annual maximum wave height and strengths of materials, describe special types of hazard functions, study positively skewed and heavy tailed data sets and provide better fits for survival data and waiting times than other competing models.

The article is composed of the following sections. In section 2, the pdf for GLBXII distribution is developed on the basis of the generalized log Pearson differential equation (GLPE). In section 3, GLBXII distribution is studied in terms of some structural properties, plots, sub-models and stochastic ordering. In section 4, moments, negative moments, central moments, incomplete moments, inequality measures, residual life functions and some other properties are presented. In section 5, reliability measures and uncertainty measures are studied. In section 6, GLBXII distribution is obtained from some compound scale mixture of generalized log-Weibull (GLW) distribution and gamma distribution and sized biased Erlang distribution. In section 7, characterizations of GLBXII distribution is studied through (i) Truncated moment of the log of the random variable; (ii) Truncated moment of a function (not log) of the random variable; (iii) the hazard function and (iv) certain functions of the random variable. In section 8, the potentiality of GLBXII distribution is demonstrated by its application to real data sets; parameters of GLBXII are estimated using maximum likelihood method. Goodness of fit of probability distribution through different methods is studied. Finally, in Section 9, we provide some concluding remarks.

2. Development of Glbxii Distribution

In this section, we derive the pdf of GLBXII using GLPE given by

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{1}{x}\frac{a_0+a_1(\ln x)+a_2(\ln x)^2+...+a_m(\ln x)^m}{b_0+b_1(\ln x)+b_2(\ln x)^2+...+b_n(\ln x)^n},\quad x>1,m,n=1,2... \qquad (3)$$

By taking $a_2=a_3=...a_{m-2}=0, b_0=0,\; b_2=b_3=...\mathrm{b}_{n-1}=0$ and $m=n=2a+1$, we have

$$\frac{d}{dx}[\ln f]=\frac{1}{x}\left\{\frac{a_0+a_1(\ln x)+a_{m-1}(\ln x)^{2a}+a_m(\ln x)^{2a+1}}{b1(\ln x)+b_n(\ln x)^{2a+1}}\right\},\; x>1. \qquad (4)$$

$For\;\; a_1=-b_1, a_m=-b_n$, after simplification and integration of both sides of (4), we have

$$f(x)=c\left(\frac{1}{x}\right)\left(\frac{\ln x}{(b_1)^{\frac{1}{2a}}}\right)^{\frac{a_0}{b_1}}\left(1+\frac{b_n}{b_1}(\ln x)^{2a}\right)^{-\left(\frac{a_0 b_n}{2ab_n b_1}-\frac{a_{m-1}}{2ab_n}\right)},\quad x>1, \qquad (5)$$

where $B(,)$ the beta function. The normalizing constant c is

$$c=\frac{2a(b_1)^{\frac{a_0}{2ab_1}}}{\left(\frac{b_1}{b_n}\right)^{\frac{a_0}{2ab_1}+\frac{1}{2a}} B\left(\frac{a_0}{2ab_1}+\frac{1}{2a},-\frac{a_{m-1}}{2ab_n}-\frac{1}{2a}\right)},\; with\;\; a_{m-1}<0.$$

Taking

$a_0=(2a-1)b^{2a},\; a_1=-b^{2a},\; a_{m-1}=-(2ap+1),\; a_m=-1,\; b_1=b^{2a}\;\; and\;\; b_n=1\;\; in\;\; (5)$, so the pdf is written as

$$f(x)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{(2a-1)}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}, x>1, a>0, b>0, p>0. \qquad (6)$$

The cdf of the random variable X with GLBXII distribution and parameters a, b *and* p is

$$F(x)=1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p},\; x\geq 1, a>0, b>0, \mathrm{p}>0. \qquad (7)$$

3. Mathematical Properties Of Glbxii Distribution

In this section, the GLBXII distribution is studied in terms of some structural properties, plots, sub-models and stochastic ordering.

3.1 Structural Properties of GLBXII Distribution

The survival, hazard, cumulative hazard and reverse hazard function of a random variable X with GLBXII distribution are given, respectively, by

$$S(x)=\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}, \tag{8}$$

$$h(x)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1}, \tag{9}$$

$$H(x)=\int_1^x \frac{f(u)}{S(u)}du=p\ln\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right], \tag{10}$$

and

$$\text{T(x)}=\frac{f(x)}{F(x)}=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^{-1}. \tag{11}$$

The mills ratio of GLBXII distribution is

$$m(x)=\frac{1-F(x)}{f(x)}=\frac{bx}{2ap}\left(\frac{\ln x}{b}\right)^{-2a+1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]. \tag{12}$$

The elasticity of GLBXII distribution is given by $e(x)=\frac{d\,\ln\mathrm{F}(x)}{d\ln x}=xT(x)$

$$e(x)=\frac{2ap}{b}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^{-1}. \tag{13}$$

The mode of GLBXII distribution with pdf (6) is obtained by solving $\frac{d}{dx}\left(\ln\left(f(x)\right)\right)=0, i.e.,$

$$\frac{-1}{x}+\frac{(2a-1)}{\left(\frac{\ln x}{b}\right)bx}-\frac{\frac{2a(p+1)}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}}{\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]}=0$$, for $a=\frac{1}{2}$, we obtain mode $(\hat{x})=e^{-(p+b+1)}$.

3.2 Quantile function of GLBXII Distribution

The quantile function of GLBXII distribution is $x_q = \exp\left(b\left[(1-q)^{-\frac{1}{p}}-1\right]^{\frac{1}{2a}}\right)$. For $q=\dfrac{1}{2}$, median of GLBXII distribution is $Median = \exp\left(b\left(2^{\frac{1}{p}}-1\right)^{\frac{1}{2a}}\right)$ and random number generator for GLBXII distribution is $X = \exp\left\{b\left[\left(\dfrac{1}{1-U}\right)^{\frac{1}{p}}-1\right]^{\frac{1}{2a}}\right\}$, where the random variable U has the uniform distribution on (0,1).

3.3 Plots of the GLBXII Density and Hazard Rate Function

The following graphs show that shapes of GLBXII density are arc, exponential, positively skewed and symmetrical. The GLBXII distribution has increasing, decreasing, upside-down bathtub and constant after some time.

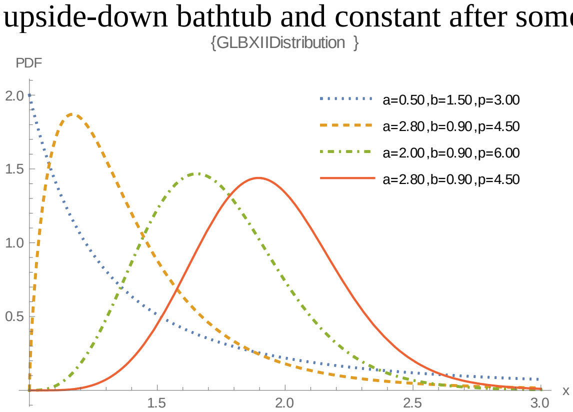

Figure 3.3a: Plots of pdf

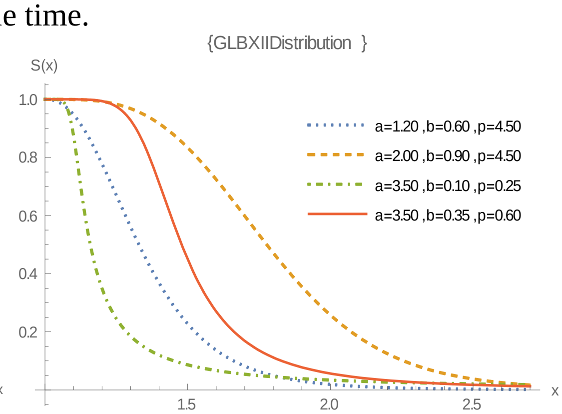

Figure 3.3b: Plots of S(x)

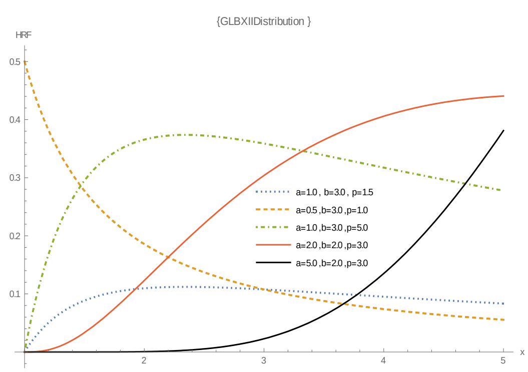

Figure 3.3c: Plots of HRF

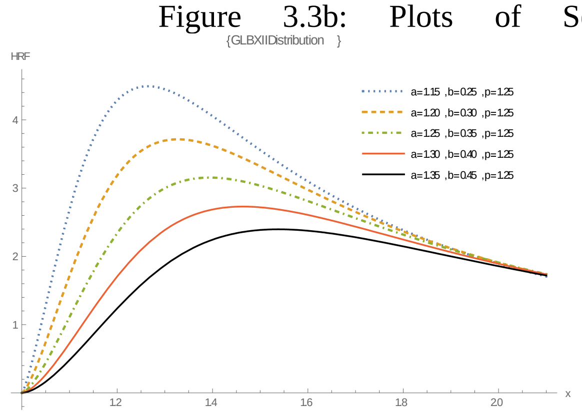

Figure 3.3b: Plots of HRF

3.4 Sub Models of GLBXII Distribution

i) For $p=1$, GLBXII distribution becomes log-Fisk distribution.

ii) For $2a=p$, GLBXII distribution becomes log Para-logistic distribution.

iii) For $a=\frac{1}{2}, p=1,$ GLBXII distribution becomes log-Inverse Pareto distribution.

iv) For $2a=\frac{1}{c}, p=1$, GLBXII distribution becomes log Pareto-III distribution.

v) For $b=1$, GLBXII distribution becomes log Burr XII (LBXII) distribution.

vi) For $a=\frac{1}{2}$, GLBXII distribution becomes log Lomax(LL) distribution.

vii) For $p\to\infty, b=\beta p^{\frac{1}{2a}}$, GLBXII distribution becomes GLW distribution.

viii) For $p\to\infty, b=\beta p^{\frac{1}{2a}}, a=\frac{1}{2}$, GLBXII distribution becomes log-exponential distribution.

ix) For $\ln X=Y$, GLBXII distribution becomes generalized Burr XII (GBXII) distribution.

x) For $\ln X=Y, b=1$, GLBXII distribution becomes BXII distribution.

xi) For $\ln X=Y, a=\frac{1}{2}$, GLBXII distribution becomes Lomax distribution.

3.5 Stochastic Ordering

Stochastic orders are widely used in Economics, Reliability, Survival Analysis and Operations Research. Stochastic orders are used to study location and magnitude of random variables.

3.5.1 Definition: A random variable X is said to be smaller than a random variable Y in the (i) stochastic order $X\le_{st} Y \text{ if } F_X(x)\ge F_Y(x)$ for all x, (ii) likelihood ratio order $X\le_{lr} Y \text{ if } \frac{f_X(x)}{f_Y(x)}$ decreases in x, (iii) mean residual life order $X\le_{mrl} Y \text{ if } m_X(x)\ge m_Y(x)$ for all x and (iv) hazard rate order $X\le_{hr} Y \text{ if } h_X(x)\ge h_Y(x)$ for all x.

Theorem 3.1:
Let $X\sim GLBXII(a_1,b_1,p_1)$ *and* $Y\sim GLBXII(a_2,b_2,p_2)$. *If* $a_1=a_2, b_1=b_2, p_1>p_2$ *or* $(b_1=b_2, p_1=p_2, a_1<a_2)$, then GLBXII distribution is ordered strongly according to likelihood ratio ordering.

Proof: For $X\sim GLBXII(a_1,b_1,p_1)$ and $Y\sim GLBXII(a_2,b_2,p_2)$ we have

$$\ln\frac{f_X(x)}{f_Y(x)}=\ln\left(\frac{a_1p_1(b_2)^{2a_2}}{a_2p_2(b_1)^{2a_1}}\right)+2(a_1-a_2)\ln(\ln x)-(p_1+1)\ln\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right]+(p_2+1)\ln\left[1+\left(\frac{\ln x}{b_2}\right)^{2a_2}\right],$$

$$\frac{d}{dx}\left[\ln\frac{f_X(x)}{f_Y(x)}\right]=\left[\frac{2(a_1-a_2)}{x\ln x}-(p_1+1)\frac{\frac{2a_1}{xb_1}\left(\frac{\ln x}{b_1}\right)^{2a_1-1}}{\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right]}+(p_2+1)\frac{\frac{2a_2}{xb_2}\left(\frac{\ln x}{b_2}\right)^{2a_2-1}}{\left[1+\left(\frac{\ln x}{b_2}\right)^{2a_2}\right]}\right],$$

Case (i) $b_1 = b_2, p_1 = p_2, a_1 < a_2$, we have $\frac{d}{dx}\left[\ln \frac{f_X(x)}{f_Y(x)}\right] < 0.$

Case (ii) $a_1 = a_2, b_1 = b_2, p_1 > p_2$, we have $\frac{d}{dx}\left[\ln \frac{f_X(x)}{f_Y(x)}\right] = [p_2 - p_1]\frac{\frac{2a_1}{xb_1}\left(\frac{\ln x}{b_1}\right)^{2a_1-1}}{\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right]}.$

So we have $\frac{d}{dx}\left[\ln \frac{f_X(x)}{f_Y(x)}\right] < 0$. Therefore for GLBXII, random variable X is said to be smaller than a random variable Y in likelihood ratio order $X \le_{lr} Y$, since $\frac{d}{dx}\left[\ln \frac{f_X(x)}{f_Y(x)}\right] < 0$.

Theorem 3.2:

Let $X \sim GLBXII(a_1, b_1, p_1)$ and $Y \sim GLBXII(a_2, b_2, p_2)$. If $a_1 = a_2, b_1 = b_2, p_1 > p_2$ or $(b_1 = b_2, p_1 = p_2, a_1 < a_2)$, then GLBXII distribution is ordered strongly according to hazard rate ordering.

Proof: For $X \sim GLBXII(a_1, b_1, p_1)$ and $Y \sim GLBXII(a_2, b_2, p_2)$ we have

$$h_X(x) = \frac{2a_1 p_1}{b_1 x}\left(\frac{\ln x}{b_1}\right)^{2a_1-1}\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right]^{-1} \quad \text{and}$$

$$h_Y(x) = \frac{2a_2 p_2}{b_2 x}\left(\frac{\ln x}{b_2}\right)^{2a_2-1}\left[1+\left(\frac{\ln x}{b_2}\right)^{2a_2}\right]^{-1},$$

$$\ln\frac{h_X(x)}{h_Y(x)} = \ln\left(\frac{a_1 p_1 (b_2)^{2a_2}}{a_2 p_2 (b_1)^{2a_1}}\right) + 2(a_1 - a_2)\ln(\ln x) - \ln\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right] + \ln\left[1+\left(\frac{\ln x}{b_2}\right)^{2a_2}\right],$$

$$\frac{d}{dx}\left[\ln\frac{h_X(x)}{h_Y(x)}\right] = 2\left[a_1\left(\frac{1}{x\ln x} - \frac{\frac{1}{xb_1}\left(\frac{\ln x}{b_1}\right)^{2a_1-1}}{\left[1+\left(\frac{\ln x}{b_1}\right)^{2a_1}\right]}\right) - a_2\left(\frac{1}{x\ln x} - \frac{\frac{1}{xb_2}\left(\frac{\ln x}{b_2}\right)^{2a_2-1}}{\left[1+\left(\frac{\ln x}{b_2}\right)^{2a_2}\right]}\right)\right],$$

Case (i) $b_1 = b_2, p_1 = p_2, a_1 < a_2$, we have $\frac{d}{dx}\left[\ln \frac{h_X(x)}{h_Y(x)}\right] < 0.$

Case (ii) $b_1 = b_2, p_1 = p_2, a_1 > a_2$, we have $\frac{d}{dx}\left[\ln \frac{h_X(x)}{h_Y(x)}\right] > 0.$

Therefore for GLBXII, random variable X is said to be smaller than a random variable Y in hazard rate order $X \le_{hr} Y$, since $h_X(x) \ge h_Y(x)$ for all x.

4. Moments

Moments, negative moments, central moments, incomplete moments, inequality measures, residual life functions and some other properties are achieved.

4.1 Moments of GLBXII Distribution

The r[th] moment about the origin of X with GLBXII distribution is

$$\mu_r' = E(X^r) = \int_1^\infty x^r f(x)dx,$$

$$\mu_r' = E(X^r) = \int_1^\infty x^r \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p+1)} dx,$$

Let $\left(\frac{\ln x}{b}\right)^{2a} = y,\ x = e^{by^{\frac{1}{2a}}}$, we have

$$E\left(X^r\right) = p\int_0^\infty \left(e^{by^{\frac{1}{2a}}}\right)^r (1+y)^{-(p+1)}dy,$$

$$p\int_0^\infty e^{rby^{\frac{1}{2a}}}(1+y)^{-(p+1)}dy = p\sum_{i=0}^\infty \frac{(rb)^i}{i!}\int_0^\infty y^{\frac{i}{2a}}(1+y)^{-(p+1)}dy,$$

$$\mu_r' = p\sum_{i=0}^\infty \frac{(rb)^i}{i!}B(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}),\ \mathrm{r}=1,2,3,... \qquad (14)$$

The mean and variance of GLBXII distribution are

$$E(X) = p\sum_{i=0}^\infty \frac{b^i}{i!}B(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}), \qquad (15)$$

$$\mathrm{Var}(X) = p\sum_{i=0}^\infty \frac{(2b)^i}{i!}B(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}) - \left(p\sum_{i=0}^\infty \frac{b^i}{i!}B(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a})\right)^2. \qquad (16)$$

The fractional positive moments of X with GLBXII distribution are

$$\mu_{\frac{m}{n}}' = E\left(X^{\frac{m}{n}}\right) = p\sum_{i=1}^\infty \frac{\left(\frac{m}{n}b\right)^i}{i!}B\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right). \qquad (17)$$

The r[th] negative moment about the origin of X with GLBXII distribution is

$$\mu_{-r}' = E\left(X^{-r}\right) = \int_1^\infty x^{-r} \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p+1)} dx,$$

$$\mu_{-r}' = p\sum_{i=1}^\infty \frac{(-rb)^i}{i!}B\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right). \qquad (18)$$

Negative moments help to harmonic mean and many other measures. The fractional negative moment of X with GLBXII distribution is

$$\mu'_{-\frac{m}{n}} = E\left(X^{-\frac{m}{n}}\right) = E\left(X^{-\frac{m}{n}}\right) = p\sum_{i=1}^{\infty}\frac{\left(-\frac{m}{n}b\right)^{i}}{i!}B\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right). \qquad (19)$$

The k^th moment about mean of X is determined from the relationship

$$\mu_k = E\left[X - E(X)\right]^k = \sum_{j=1}^{k}\binom{k}{j}(-1)^j \mu'_j \mu'_{(k-j)}, \quad (20)$$

The kth moment about mean of X with GLBXII distribution is determined from the relationship

$$\mu_k = \sum_{j=1}^{k}\binom{k}{j}(-1)^j\left\{\left[p\sum_{i=0}^{\infty}\frac{(jb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\left[p\sum_{i=0}^{\infty}\frac{((k-j)b)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\right\} \qquad (21)$$

Pearson's measure of skewness γ_1 is given by $\gamma_1 = \dfrac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$,

$$\gamma_1 = \frac{\sum_{j=1}^{3}\binom{3}{j}(-1)^j\left\{\left[p\sum_{i=0}^{\infty}\frac{(jb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\left[p\sum_{i=0}^{\infty}\frac{((3-j)b)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\right\}}{\left(\sum_{j=1}^{2}\binom{2}{j}(-1)^j\left\{\left[p\sum_{i=0}^{\infty}\frac{(jb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\left[p\sum_{i=0}^{\infty}\frac{((2-j)b)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\right\}\right)^{\frac{3}{2}}}. \qquad (22)$$

Pearson measure of Kurtosis β_2 is given by $\beta_2 = \dfrac{\mu_4}{(\mu_2)^2}$,

$$\beta_2 = \frac{\sum_{j=1}^{4}\binom{4}{j}(-1)^j\left\{\left[p\sum_{i=0}^{\infty}\frac{(jb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\left[p\sum_{i=0}^{\infty}\frac{((4-j)b)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\right\}}{\left(\sum_{j=1}^{2}\binom{2}{j}(-1)^j\left\{\left[p\sum_{i=0}^{\infty}\frac{(jb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\left[p\sum_{i=0}^{\infty}\frac{((2-j)b)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right)\right]\right\}\right)^{2}}. \qquad (23)$$

The moment generating function for the random variable X having GLBXII distribution is

$$M_X(t) = E\left[e^{tX}\right] = \sum_{r=1}^{\infty}\frac{t^r}{\text{r}!}E\left(X^r\right) = \sum_{r=1}^{\infty}\frac{t^r}{\text{r}!}\mu'_r = \sum_{r=1}^{\infty}\sum_{i=0}^{\infty}\frac{t^r}{\text{r}!}\frac{(rb)^i}{i!}B\left(\frac{i}{2a}+1,\text{p}-\frac{i}{2a}\right). \qquad (24)$$

The cumulants are obtained from the recurrence relation $k_r = E\left(X^r\right) - \sum_{c=1}^{r-1}\binom{r-1}{c-1}k_c E\left(X^{r-c}\right)$.

The cumulants of GLBXII distribution are obtained from the recurrence relation

$$k_r = \sum_{i=0}^{\infty} \frac{(rb)^i}{i!} B\left(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}\right) - \sum_{c=1}^{r-1} \binom{r-1}{c-1} k_c \sum_{i=0}^{\infty} \frac{((r-c)b)^i}{i!} B\left(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}\right). \qquad (25)$$

The r^{th} moment about the origin of ln(X) with GLBXII distribution is

$$E\left((\ln(\mathrm{X}))^r\right) = \int_1^{\infty} (\ln x)^r \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p+1)} dx, \qquad (26)$$

Let $\left(\frac{\ln x}{b}\right)^{2a} = y, \quad \ln x = b y^{\frac{1}{2a}}$, we have

$$E\left((\ln(\mathrm{X}))^r\right) = p b^r \int_0^{\infty} y^{\frac{r}{2a}} (1+y)^{-(p+1)} dy,$$

$$E\left((\ln(\mathrm{X}))^r\right) = p b^r B\left(\frac{r}{2a}+1, p-\frac{r}{2a}\right).$$

The factorial moments for GLBXII distribution are given by

$$E[X]_n = \sum_{r=1}^{n} \alpha_r E(X^r) = \sum_{r=1}^{n} \alpha_r \mu_r',$$

$$E[X]_n = \sum_{r=1}^{n} \alpha_r \mu' = \sum_{r=1}^{n}\left[\alpha_r p \sum_{i=0}^{\infty} \frac{(rb)^i}{i!} B\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right)\right], \qquad (27)$$

$where\ [X]_i = X(X+1)(X+2)......(X+i-1)$ and α_r is Stirling number of the first kind.

4.2 The Mellin Transform of GLBXII Distribution

The Mellin transform helps to determine moments for a probability distribution. By definition, the Mellin transform is

$$\mathrm{M}\{f(x);k\} = f^*(k) = \int_1^{\infty} f(x)\, x^{k-1} dx.$$

The Mellin transform of X with GLBXII distribution is written as

$$\mathrm{M}\{f(x);k\} = \int_1^{\infty} x^{k-1} \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p+1)} dx,$$

$$\mathrm{M}\{f(x);k\} = p\sum_{i=1}^{\infty} \frac{((k-1)b)^i}{i!} B\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right). \qquad (28)$$

4.3 Moments of Order Statistics

Moments of order statistics have wide applications in reliability and life testing. Moments of order statistics also design to replacement policy with the prediction of failure of future items determined from few early failures.

The pdf for m^{th} order statistic $X_{m:n}$ is

$$f(x_{m:n}) = \frac{1}{B(m, n-m+1)} \left[F(x)\right]^{m-1} \left[1-F(x)\right]^{n-m} f(x). \qquad (29)$$

The pdf of m^{th} order statistic $X_{m:n}$ for GLBXII distribution is

$$f(x_{m:n}) = \frac{\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j}}{B(m, n-m+1)} \frac{2ap}{bx} \left(\frac{\ln x}{b}\right)^{2a-1} \left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+p(n-m)+pj+1)}. \qquad (30)$$

The r^th moments about the origin of m^{th} order statistic $X_{m:n}$ for GLBXII distribution is given by

$$E(X^r_{m:n}) = \int_1^\infty x^r \frac{\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j}}{B(m, n-m+1)} \frac{2ap}{bx} \left(\frac{\ln x}{b}\right)^{2a-1} \left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+p(n-m)+pj+1)} dx,$$

$$E(X^r_{m:n}) = \int_1^\infty x^r \frac{\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j}}{B(m, n-m+1)} \frac{2ap}{bx} \left(\frac{\ln x}{b}\right)^{2a-1} \left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+p(n-m)+pj+1)} dx,$$

Let $\left(\frac{\ln x}{b}\right)^{2a} = y$, $x = e^{by^{\frac{1}{2a}}}$, we have

$$E(X^r_{m:n}) = \frac{\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j}}{B(m, n-m+1)} p \sum_{i=0}^{\infty} \frac{(rb)^i}{i!} B\left(\frac{i}{2a}+1, \mathrm{p}(n+j-m+1) - \frac{i}{2a}\right). \qquad (31)$$

Mean of m^{th} order statistic $X_{m:n}$ for GLBXII distribution is given as

$$E(X_{m:n}) = \frac{\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j}}{B(m, n-m+1)} p \sum_{i=0}^{\infty} \frac{b^i}{i!} B\left(\frac{i}{2a}+1, \mathrm{p}(n+j-m+1) - \frac{i}{2a}\right). \qquad (32)$$

4.4 L-Moments and TL-Moments

L-moments (Hosking; 1990) are used to estimate the parameters. The r^th L-moment for a probability distribution is

$$\lambda_{r+1} = \frac{1}{(r+1)} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k:r+1}) \quad r = 0,1,2....,$$

$$\lambda_{r+1} = \frac{1}{(r+1)} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \int x f_{r+1-k:r+1}(x)\, dx,$$

$$\lambda_{r+1} = \sum_{k=0}^{r} (-1)^k \binom{r}{k}^2 \int x \left[F(x)\right]^{r-k} \left[1-F(x)\right]^k f(x)\, dx.$$

The Legendre polynomial is

$$P_r(F(x)) = \sum_{k=0}^{r} (-1)^k \binom{r}{k}^2 \left[F(x)\right]^{r-k} \left[1-F(x)\right]^k,$$

and the rth shifted polynomial is

$$P_r^*(F(x)) = P_r(2F(x)-1) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\binom{r+k}{k}[F(x)]^k \ ,$$

According to Hosking (1990), the rth L-moments for a probability distribution are

$$\lambda_{r+1} = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\binom{r+k}{k}\int x[F(x)]^k f(x)dx,$$

The rth L-moments for a GLBXII distribution are

$$\lambda_{r+1} = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\binom{r+k}{k}\int_1^\infty x\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^k \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)} dx$$

$$\lambda_{r+1} = p\sum_{k=0}^{r}\sum_{j=0}^{k}\sum_{i=0}^{\infty}(-1)^{r-k+j}\binom{k}{j}\binom{r}{k}\binom{r+k}{k}\frac{b^i}{i!}B\left(\frac{i}{2a}+1, \mathrm{p}(j+1)-\frac{i}{2a}\right). \qquad (33)$$

The probability weighted moments (PWM) for GLBXII distribution are

$$l_k = \int_1^\infty x\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^k \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)} dx,$$

$$l_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\int_1^\infty x\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(pj+p+1)} dx,$$

$$l_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}p\sum_{i=0}^{\infty}\frac{b^i}{i!}B\left(\frac{i}{2a}+1, \mathrm{p}(j+1)-\frac{i}{2a}\right), k=0,1,2,3\ldots \ . (34)$$

TL-moments (Elamir and Seheult 2003) are stronger than L-moments due to trimming of outliers. TL-moments provide best estimates of the parameters for the probability distributions.

The rth TL-moment for a probability distribution is defined as

$$\lambda_r^{(1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}E\left(X_{r+k+1\,:\,r+2}\right),$$

where $E\left(X_{r+k+1\,:\,r+2}\right) = (r+k+1)\binom{r+k+1}{r+2}\int x(F(x))^{r+k+1}(1-F(x))^{k-1}dF(x)$.

The rth TL-moments for GLBXII distribution is given by

$$E\left(X_{r+k+1\,:\,r+2}\right) = (r+k+1)\binom{r+k+1}{r+2}\int_1^\infty x\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^{r+k+1}\left(1-1+\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\right)^{k-1}\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)} dx$$

$$E\left(X_{r+k+1\,:\,r+2}\right) = (r+k+1)\binom{r+k+1}{r+2}\sum_{l=0}^{r+k+1}(-1)^l\binom{r+k+1}{l}\int_1^\infty x\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p(r+2k+1-l)-1} dx$$

Let $y=\left(\frac{\ln x}{b}\right)^a, x=e^{by^{1/a}}, dy=\frac{a}{bx}\left(\frac{\ln x}{b}\right)^{a-1}dx,$ we have

$$E\left(X_{r+k+1\,:\,r+2}\right)=\left(r+k+1\right)\binom{r+k+1}{r+2}\sum_{l=0}^{r+k+1}\left(-1\right)^{l}\binom{r+k+1}{l}\sum_{j=0}^{\infty}\frac{b^{j}}{j!}\int_{0}^{\infty}y^{\frac{j}{2a}+1-1}\left[1+y\right]^{-p(r+2k+1-l)-1}dy,$$

$$E\left(X_{r+k+1\,:\,r+2}\right)=\left(r+k+1\right)\binom{r+k+1}{r+2}\sum_{l=0}^{r+k+1}\left(-1\right)^{l}\binom{r+k+1}{l}\sum_{j=0}^{\infty}\frac{b^{j}}{j!}B\left(\frac{j}{2a}+1,\mathrm{p}\left(r+2k+1-l\right)-\frac{j}{2a}\right),$$

$$\lambda_{r}^{(1)}=\frac{1}{r}\sum_{k=0}^{r-1}\left(-1\right)^{k}C_{k}^{r-1}\left\{\left(r+k+1\right)\binom{r+k+1}{r+2}\sum_{l=0}^{r+k+1}\left(-1\right)^{l}\binom{r+k+1}{l}p\sum_{j=0}^{\infty}\frac{b^{j}}{j!}B\left(\frac{j}{2a}+1,\mathrm{p}\left(r+2k+1-l\right)-\frac{j}{2a}\right)\right\}. \quad (35)$$

4.5 Incomplete Moments

Incomplete moments are used in mean inactivity life, mean residual life function, and other inequality measures. The lower incomplete moments for random variable X having GLBXII distribution are

$$M_{r}'\left(z\right)=E_{X\le z}\left(X^{r}\right)=\int_{1}^{z}x^{r}f\left(x\right)dx=\int_{1}^{z}x^{r}\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p+1)}dx,$$

Let $\left(\frac{\ln x}{b}\right)^{2a}=y, \quad x=e^{by^{\frac{1}{2a}}}$, we have

$$M_{r}'\left(z\right)=p\int_{0}^{\left(\frac{\ln z}{b}\right)^{2a}}\left(e^{by^{\frac{1}{2a}}}\right)^{r}\left(1+y\right)^{-(p+1)}dy,$$

$$M_{r}'\left(z\right)=p\sum_{i=1}^{\infty}\frac{\left(rb\right)^{i}}{i!}\int_{0}^{\left(\frac{\ln z}{b}\right)^{2a}}y^{\frac{i}{2a}}\left(1+y\right)^{-(p+1)}dy,$$

$$M_{r}'\left(z\right)=p\sum_{i=1}^{\infty}\frac{\left(rb\right)^{i}}{i!}B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1,p-\frac{i}{2a}\right), \quad (36)$$

where $B_{x}\left(-\right)$ is incomplete beta function.

The upper incomplete moments for random variable X having GLBXII distribution are

$$E_{X>z}\left(X^{r}\right)=\int_{z}^{\infty}x^{r}f\left(x\right)dx=\int_{1}^{\infty}x^{r}f\left(x\right)dx-\int_{1}^{z}x^{r}f\left(x\right)dx$$

$$E_{X>z}\left(X^{r}\right)=\mu_{r}'-M_{r}'\left(z\right),$$

From (14) and (36), we have

$$E_{X>z}\left(X^{r}\right)=p\sum_{i=1}^{\infty}\frac{\left(rb\right)^{i}}{i!}\left[B\left(\frac{i}{2a}+1,\mathrm{p}-\frac{i}{2a}\right)-B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1,p-\frac{i}{2a}\right)\right]. \quad (37)$$

The mean deviation about mean is $MD_{\bar{X}} = E\left|X-\mu_1^1\right| = 2\mu_1^1 F\left(\mu_1^1\right) - 2\mu_1^1 M'\left(\mu_1^1\right)_1$, and mean deviation about median is $MD_M = E\left|X-M\right| = 2MF\left(M\right) - 2MM_1'\left(M\right)$ where $\mu_1^/ = E\left(X\right)$ and
$M = Q\left(0.5\right)$. Bonferroni and Lorenz curves for a specified probability p are computed by $\mathrm{B}(p) = M_1^/\left(q\right)/p\mu_1^1$ and $L(p) = M_1^/\left(q\right)/\mu_1^1$, where $q = Q(p)$.

4.6 Residual Life functions

The residual life, say, $m_n\left(z\right)$ of X with GLBXII distribution has the following nth moment

$$m_n\left(z\right) = E\left[\left(X-z\right)^n / X > z\right] = \frac{1}{S\left(z\right)}\int_z^\infty \left(x-z\right)^n f(x)dx,$$

$$m_n\left(z\right) = \frac{1}{\left[1-F\left(z\right)\right]}\sum_{r=0}^{n}\binom{n}{r}\left(-z\right)^{n-r}\int_z^\infty x^r f\left(x\right)dx,$$

$$m_n\left(z\right) = \left[1-F\left(z\right)\right]^{-1}\sum_{r=0}^{n}\binom{n}{r}\left(-z\right)^{n-r} E_{X>z}\left(X^r\right).$$

From above equation and (37), we have

$$m_n\left(z\right) = \frac{1}{\left(1-F\left(z\right)\right)}\sum_{r=0}^{n}\binom{n}{r}\left(-z\right)^{n-r} p\sum_{i=1}^{\infty}\frac{\left(rb\right)^i}{i!}\left[B\left(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}\right) - B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right)\right]. \quad (38)$$

The average remaining lifetime of a component at time z, say, $m_1\left(z\right)$ or life expectancy is known as mean residual life (MRL) function is given by

$$m_1\left(z\right) = \frac{1}{\left(1-F\left(z\right)\right)}\sum_{r=0}^{1}\binom{1}{r}\left(-z\right)^{1-r} p\sum_{i=1}^{\infty}\frac{\left(rb\right)^i}{i!}\left[B\left(\frac{i}{2a}+1, \mathrm{p}-\frac{i}{2a}\right) - B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right)\right]. \quad (39)$$

The reverse residual life, say, $M_n\left(z\right)$ of X with GLBXII distribution having nth moment is

$$M_n\left(z\right) = E\left[\left(z-X\right)^n / X \le z\right] = \frac{1}{F\left(z\right)}\int_1^z \left(z-x\right)^n f(x)dx,$$

$$M_n\left(z\right) = \frac{1}{F\left(z\right)}\sum_{r=0}^{n}\left(-1\right)^r\binom{n}{r}z^{n-r}\int_1^z x^r f\left(x\right)dx = \frac{1}{F\left(z\right)}\sum_{r=0}^{n}\left(-1\right)^r\binom{n}{r}z^{n-r}E_{X\le z}\left(X^r\right).$$

From above equation and (36), we have

$$M_n\left(z\right) = \frac{1}{F\left(z\right)}\sum_{r=0}^{n}\left(-1\right)^r\binom{n}{r}z^{n-r} p\sum_{i=1}^{\infty}\frac{\left(rb\right)^i}{i!}B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1, p-\frac{i}{2a}\right). \quad (40)$$

The waiting time z for failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity time. The waiting time z for failure of a component of X having GLBXII distribution is defined by $M_1(z)=\frac{1}{F(z)}\sum_{r=0}^{1}(-1)^r\binom{1}{r}z^{1-r}p\sum_{i=1}^{\infty}\frac{(rb)^i}{i!}B_{\left(\frac{\ln z}{b}\right)^{2a}}\left(\frac{i}{2a}+1,p-\frac{i}{2a}\right).$ (41)

The median inactivity time function in terms of cdf of a continuous life time distribution is

$$MDIT(z)=z-F_X^{-1}\left(\frac{1}{2}F_X(z)\right).$$

The median inactivity time function in terms of cdf of GLBXII distribution is written as

$$MDIT(z)=z-\exp\left(b\left[\left[1-\frac{1}{2}\left(1-\left[1+\left(\frac{\ln z}{b}\right)^{2a}\right]^{-p}\right)\right]^{-\frac{1}{p}}-1\right]^{\frac{1}{2a}}\right). \tag{42}$$

5. Reliability and Uncertainty Measures

In this section, reliability and uncertainty measures are studied.

5.1 Stress-strength Reliability for GLBXII Distribution

If $X_1 : GLBXII(a,b,p_1), X_2 : GLBXII(a,b,p_2)$ such that X_1 represents "strength" and X_2 represents "stress" and X_1, X_2 follow a joint pdf $f(x_1,x_2)$, then $R=\Pr(X_2<X_1)$ is reliability parameter. Thus $R=\Pr(X_2<X_1)=\int_1^{\infty}f_{x_1}(x)F_{x_2}(x)dx$ is the characteristic of the distribution of X_1 and X_2. The reliability of the component is computed from GLBXII distribution as

$$R=\int_1^{\infty}f_{x_1}(x)F_{x_2}(x)dx$$

$$R=\int_1^{\infty}\frac{2ap_1}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p_1+1)}\left[1-\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-p_2}\right]dx$$

$$R=1-\int_1^{\infty}\frac{2ap_1}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{2a}\right)^{-(p_1+p_2+1)}dx$$

$$R=\frac{p_2}{(p_1+p_2)}. \tag{43}$$

Therefore (i) R is independent of a and b. (ii) for $p_1=p_2$, R=0.5. It means that X_1 and X_2 are independently identically distributed (i.i.d) and there is equal chance that X_1 is bigger than X_2.

5.2 Estimation of Multicomponent Stress-Strength system Reliability for GLBXII Distribution

Suppose a machine has at least "s" components working out of "k" component. The strengths of all components of system are $X_1, X_2, ...X_k$ and stress Y is applied on system. Both strengths $X_1, X_2, ...X_k$ and stress Y are i.i.d. distributed. G is cdf of Y and F is cdf of X. The reliability of a machine is the probability that the machine functions properly i.e.

$$R_{s,k} = P(strengths > stress) = P[atleast\ "s"\ of\ (X_1, X_2, ...X_k)\ exceed\ Y] \ldots\ldots \quad (44)$$

$If \quad X : GLBXII(a,b,p_1)\ and\ Y : GLBXII(a,b,p_2)$ **with unknown shape parameters** $p_1,\ p_2$

and common scale parameter b, where $X\ and\ Y$ and are independently distributed. The reliability in multicomponent stress- strength for GLBXII distribution using:

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{-\infty}^{\infty} [1-F(y)]^{l} [F(y)]^{k-l}\, dG(y) \quad \text{(Bhattacharyya and Johnson; 1974)} \quad (45)$$

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{1}^{\infty} \left(\left[1 + \left(\frac{\ln y}{b} \right)^{2a} \right]^{-p_1} \right)^{l} \left(1 - \left[1 + \left(\frac{\ln y}{b} \right)^{2a} \right]^{-p_1} \right)^{(k-l)} \frac{2ap_2}{by} \left(\frac{\ln y}{b} \right)^{2a-1} \left[1 + \left(\frac{\ln y}{b} \right)^{2a} \right]^{-(p_2+1)} dy,$$

$$Let\ \ t = \left[1 + \left(\frac{\ln y}{b} \right)^{2a} \right]^{-p_2},\ dt = -\frac{2ap_2}{by} \left(\frac{\ln y}{b} \right)^{2a-1} \left[1 + \left(\frac{\ln y}{b} \right)^{2a} \right]^{-(p_2-1)} dy,$$

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{0}^{1} \left(t^{\nu} \right)^{l} \left(1 - t^{\nu} \right)^{(k-l)} dt,$$

Again taking $z = t^{\nu}$, we arrive at

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{0}^{1} (z)^{l} (1-z)^{(k-l)} \frac{1}{\nu} z^{\frac{1}{\nu}-1} dz \quad where\ \nu = \frac{p_1}{p_2},$$

$$R_{s,k} = \frac{1}{\nu} \sum_{l=s}^{k} \binom{k}{l} B\left(l + \frac{1}{\nu}, k - l + 1 \right) = \frac{1}{\nu} \sum_{l=s}^{k} \frac{k!}{(k-l)!} \left(\prod_{j=0}^{l} (k + \nu - j) \right)^{-1}. \quad (46)$$

The probability $R_{s,k}$ in (46) is called multicomponent stress-strength model reliability.

5.3 Shannon Entropy and Awad Entropy

According to Shannon (1948), the measurement of expected information in a message is called entropy. Shannon entropy for random variable X with pdf (6) is given as

$$h(X) = E(-\ln f(x)) = -\int f(x) \ln f(x)\, dx \qquad (47)$$

Shannon entropy for GLBXII random variable X with pdf (6) is given by

$$h(X)=-\int_1^\infty \ln\left\{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right\}\left(\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right)dx,$$

$$h(X)=\left[pbB\left(\frac{1}{2a}+1,p-\frac{1}{2a}\right)-\ln\left(\frac{2ap}{b}\right)+\frac{(2a-1)}{2a}(\psi(p)-\gamma)-\frac{(p+1)}{p}\right], \qquad (48)$$

where $\psi(x)$ is digamma function, $\psi(p)=\frac{d}{dp}\left[\ln\Gamma(p)\right]$ and $\psi(1)=\gamma$.

Awad (1987) provided the extension of Shannon entropy as

$$A(X)=-\int_1^\infty f(x)\ln\frac{f(x)}{\delta}dx,$$

where δ is maximum value of ordinate of GLBXII distribution in the domain of X.
If random variable X has GLBXII distribution, then Awad entropy is given by

$$A(X)=\ln\delta+pbB\left(\frac{1}{2a}+1,p-\frac{1}{2a}\right)-\ln\left(\frac{2ap}{b}\right)+\frac{(2a-1)}{2a}(\psi(p)-\gamma)-\frac{(p+1)}{p}. \qquad (49)$$

5.4 Rényi Entropy, Q-Entropy, Havrda and Chavrat Entropy and Tsallis-Entropy

Rényi entropy (1961) is an extension of Shannon entropy. Rényi entropy for GLBXII random variable X with pdf (6) is theoretically computed as

$$I_R=\frac{1}{1-\nu}\log\left\{\int_1^\infty\left[f(x)\right]^\nu\right\}dx \;\; \nu\neq 1,\nu>0, \qquad (50)$$

$$\int_1^\infty\left[f(x)\right]^\nu dx=\int_1^\infty\left[\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right]^\nu dx=\frac{2^\nu a^\nu p^\nu}{\Gamma(\nu(p+1))}\sum_{k=0}^\infty\frac{(-1)^k\Gamma(\nu(p+1)+k)}{k!}\frac{\Gamma(2a(\nu+k)-\nu+1)}{(\nu-1)^{2a(\nu+k)-(\nu-1)}b^{2a(\nu+k)}}$$

$$I_R=\frac{1}{1-\nu}\log\left\{\frac{2^\nu a^\nu p^\nu}{\Gamma(\nu(p+1))}\sum_{k=0}^\infty\frac{(-1)^k\Gamma(\nu(p+1)+k)}{k!}\frac{\Gamma(2a(\nu+k)-\nu+1)}{(\nu-1)^{2a(\nu+k)-(\nu-1)}b^{2a(\nu+k)}}\right\}. \qquad (51)$$

The Q-entropy for GLBXII distribution is

$$H_q(f)=\frac{1}{1-q}\log\left\{1-\int_1^\infty\left[f(x)\right]^q\right\}dx \;\; q\neq 1,q>0,$$

$$H_q(f)=\frac{1}{1-q}\log\left[1-\frac{2^q a^q p^q}{\Gamma(q(p+1))}\sum_{k=0}^\infty\frac{(-1)^k\Gamma(q(p+1)+k)}{k!}\frac{\Gamma(2a(q+k)-q+1)}{(\nu-1)^{2a(q+k)-(q-1)}b^{2a(q+k)}}\right]. \qquad (52)$$

The Havrda and Chavrat entropy (1967) for GLBXII distribution is

$$S_{HC}(f)=\frac{1}{\nu-1}\log\left\{\int_1^\infty\left[f(x)\right]^\nu\right\}dx \;\; \nu\neq 1,\nu>0,$$

$$I_{HC}=\frac{1}{\nu-1}\log\left\{\frac{2^\nu a^\nu p^\nu}{\Gamma(\nu(p+1))}\sum_{k=0}^\infty\frac{(-1)^k\Gamma(\nu(p+1)+k)}{k!}\frac{\Gamma(2a(\nu+k)-\nu+1)}{(\nu-1)^{2a(\nu+k)-(\nu-1)}b^{2a(\nu+k)}}\right\}. \qquad (53)$$

The Tsallis-entropy (1988) for GLBXII distribution is

$$S_q(f)=\frac{1}{q-1}\log\left\{1-\int_1^\infty [f(x)]^q\right\}dx\ q\neq 1, q>0,$$

$$S_q(f(x))=\frac{1}{q-1}\left(1-\frac{2^\nu a^\nu p^\nu}{\Gamma(\nu(p+1))}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\nu(p+1)+k)}{k!}\frac{\Gamma(2a(\nu+k)-\nu+1)}{(\nu-1)^{2a(\nu+k)-(\nu-1)}b^{2a(\nu+k)}}\right). \qquad (54)$$

Shannon entropy, Collision entropy (quadratic entropy), Hartley entropy and Min entropy can be obtained from Rényi entropy. For $\nu\to 1$, Rényi entropy tends to Shannon entropy. $For\ \nu\to 2$, Rényi entropy tends to quadratic entropy. Entropies are applied to study heart beat intervals cardiac autonomic neuropathy (CAN), DNA sequences, anomalous diffusion, daily temperature fluctuations (climatic), and information content signals.

6. Compound Probability Distribution

In this section, GLBXII distribution is obtained from some compound scale mixture of GLW distribution and gamma distribution and sized biased Erlang distribution.

6.1 Development of Log-Weibull Distribution

If $p\to\infty, b=\beta p^{\frac{1}{2a}}$, GLBXII distribution will become GLW distribution with the pdf

$$w(x;a,\beta)=\frac{2a}{\beta x}\left(\frac{\ln x}{\beta}\right)^{2a-1}e^{-\left(\frac{\ln x}{\beta}\right)^{2a}},\quad x>1,\beta>0, a>0 \text{ and cdf}$$

$$W(x;a,\beta)=1-e^{-\left(\frac{\ln x}{\beta}\right)^{2a}},\ x\geq 1.$$

6.2 Compound Scale Mixture of GLW Distribution with Gamma Distribution

Theorem 6.1: Let $w(x,a,b/\theta)=\frac{2a\theta}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}e^{-\theta\left(\frac{\ln x}{b}\right)^{2a}}$ $x>1$, be pdf of GLW distribution and θ have gamma distribution with pdf $g(\theta,\alpha,p)=\frac{\alpha^p}{\Gamma(p)}\theta^{p-1}e^{-\alpha\theta}\quad \theta>0$. Then X has pdf (6).

Proof:

For compounding $f(x,a,b,p)=\int_0^\infty w(x,a,b/\theta)g(\theta,\alpha,p)d\theta$

$$f(x,a,b,p)=\int_0^\infty \frac{2a\theta}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}e^{-\theta\left(\frac{\ln x}{b}\right)^{2a}}\frac{1}{\Gamma(p)}\theta^{p-1}e^{-\theta}d\theta$$

$$f(x,a,b,p)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\quad x\geq 1 \text{ is pdf of GLBXII}$$

distribution.

6.3 Compound Scale Mixture Of GLW Distribution And Sized Biased Erlang Distribution.

Theorem 6.2: Let $w(x,a,b,\alpha/\theta)=\frac{2a\theta\alpha}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}e^{-\theta\alpha\left(\frac{\ln x}{b}\right)^{2a}}, x>1$ be pdf of GLW distribution and θ have sized biased moment Erlang distribution with pdf $g(\theta,\gamma,p)=\frac{\alpha^{p-\gamma}}{\Gamma(p-\gamma)}\theta^{p-\gamma-1}e^{-\alpha\theta},\ \theta>0.$ Then X has pdf (6).

Proof:

$$f(x,a,b,p)=\int_0^\infty w(x,a,b/\theta)g(\theta,\alpha,\gamma,p)d\theta$$

$$f(x,a,b,p)=\int_0^\infty \frac{2a\alpha\theta}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}e^{-\theta\alpha\left(\frac{\ln x}{b}\right)^{2a}}\frac{\alpha^{p-\gamma}}{\Gamma(p-\gamma)}\theta^{p-\gamma-1}e^{-\alpha\theta}d\theta$$

$$f(x,a,b,p,\gamma)=\frac{2a(p-\gamma)}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p-\gamma+1)}$$

For $\gamma=0$, we have $f(x,a,b,p)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}$ is pdf of GLBXII distribution.

7. Characterization

In this section, GLBXII distribution is characterized through: (i) Truncated moment of the log of random variable (ii) Truncated moment of a function (not log) of random variable; (ii) Truncated moment of a function (not log) of the random variable; (iii) the hazard function and (iv) certain functions of the random variable. One of the advantages of characterization (ii) is that the cdf is not required to have a closed form. We present our characterizations (i) - (iv) in four subsections.

7.1 Characterization Based on Truncated Moment of Log of the Random Variable

Here is our first characterization of GLBXII distribution.

Proposition 7.1.1: Let $X:\Omega\to(1,\infty)$ be a continuous random variable with cdf $F(x)$ ($0<F(x)<1$ for $x>1$), then for p>1, X has cdf (7) if and only if

$$E\left[(\ln X)^{2a}\middle| X>t\right]=\frac{1}{p-1}\left\{p(\ln(t))^{2a}+b^{2a}\right\}\text{ holds.}\tag{56}$$

Proof If X has X cdf (7), then

$$E\left[(\ln(X))^{2a}\middle| X>t\right]=(1-F(t))^{-1}\int_t^\infty(\ln x)^{2a}f(x)dx$$

$$=\left[1+\left(\frac{\ln t}{b}\right)^{2a}\right]^{p}\int_{t}^{\infty}(\ln x)^{2a}\times$$

$$\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}dx.$$

Upon integration by parts and simplification, we arrive at

$$E\left[(\ln X)^{2a}\middle|X>t\right]=\frac{1}{p-1}\left\{p(\ln(t))^{2a}+b^{2a}\right\}\qquad for\ t>1.$$

Conversely if proposition 7.1.1 holds then

$$(1-F(t))^{-1}\int_{t}^{\infty}(\ln X)^{2a}f(x)dx=\frac{b^{2a}}{p-1}\left\{p\left(\frac{\ln(t)}{b}\right)^{2a}+1\right\},$$

$$\int_{t}^{\infty}(\ln X)^{2a}f(x)dx=\frac{b^{2a}(1-F(t))}{p-1}\left\{p\left(\frac{\ln(t)}{b}\right)^{2a}+1\right\}.$$

Differentiating both sides above equation with respect to t, we obtain

$$-(\ln t)^{2a}\ f(t)=\frac{b^{2a}(1-F(t))}{p-1}\left(\frac{2ap}{bt}\left(\frac{\ln(t)}{b}\right)^{2a-1}\right)-\frac{b^{2a}f(t)}{p-1}\left(p\left(\frac{\ln(t)}{b}\right)^{2a}+1\right).$$

After simplification and integration we arrive at

$$F(t)=1-\left[1+\left(\frac{\ln t}{b}\right)^{2a}\right]^{-p}\ \text{for } t\geq 1.$$

7.2 Characterizations Based on Truncated Moment of A Function of The Random Variable

In this subsection we first present a characterization of GLBXII distribution in terms of a simple relationship between truncated moment of function of X and another function. This characterization result employs a version of the theorem due to Glänzel (1987); see Theorem 1 of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 7.2.1: Let $X:\Omega\to(1,\infty)$ be a continuous random variable and let $q(x)=\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1}$ *for* $x>1$. The random variable X has pdf (6) if and only if the function

η defined in Theorem 1 has the form $\eta(x)=\frac{p}{p+1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1},\quad x\geq 1.$

Proof: Let the random variable X with pdf (6), then

$$\left(1-F(x)\right)E\left(q(x)\middle| X\geq x\right)=\frac{p}{p+1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\quad x>1.$$

or

$$E\left(q(x)\middle| X\geq x\right)=\frac{p}{p+1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1},\ x>1.$$

and

$$\eta(x)-q(x)=-\frac{1}{p+1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1},\quad x\geq 1$$

Conversely if η is given as above, then

$$s'(x)=\frac{\eta'(x)}{\eta(x)-q(x)}=\frac{2ap\left(\frac{\ln x}{b}\right)^{2a-1}}{bx\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]},x>1,$$

and hence

$$s(x)=\ln\left\{\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{p}\right\}\ x>1,$$

and

$$e^{-s(x)}=\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-p}\quad x>1.$$

In view of Theorem 1, X has density (6).

Corollary 7.2.1: Let $X:\Omega\to(1,\infty)$ be a continuous random variable. The pdf of X is (6) if and only if there exist functions $\eta(x)$ *and* $q(x)$ defined in Theorem 1satisfying the differential

equation $$\frac{\eta'(x)}{\eta(x)-q(x)}=\frac{2ap\left(\frac{\ln x}{b}\right)^{2a-1}}{bx\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]}.$$

The general solution of differential equation in corollary is

$$\eta(x)=\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{p}\left[-\int\frac{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}}{\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{p+1}}q(x)dx+D\right]$$

where D is constant. Note that a set of functions satisfying above differential equation is given in Proposition 7.2.1 with D=0. However, it should also be noted that are other pairs (η, q) satisfying conditions in Theorem 1.

7.3 Characterization Based on Hazard Function

It is well known that hazard function, h_F of a twice differential distribution function, F, satisfies the first order differential equation $\frac{d}{dx}\left[\log f(x)\right] = \frac{h_F'(x)}{h_F(x)} - h_F(x)$.

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterizations establish a non-trivial characterization of GLBXII distribution which is not of the above trivial form.

Proposition 7.3.1: Let $X:\Omega \to (1,\infty)$ be continuous random variable .The pdf of X is (6) if and only if its hazard function, h_F satisfies the first order differential equation

$$\left(xh_F' + h_F\right) = \frac{2ap\left(\frac{\ln x}{b}\right)^{2a-2}\left[(2a-1) - \left(\frac{\ln x}{b}\right)^{2a}\right]}{b^2 x\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]^2}.$$

Proof: If X has pdf (6), then above differential equation surely holds. Now if differential equation

holds then $\frac{d}{dx}\left\{xh_F(x)\right\} = \frac{2ap}{b}\frac{d}{dx}\left[\frac{\left(\frac{\ln x}{b}\right)^{2a-1}}{\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]}\right]$,

or

$$h_F(x) = \frac{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}}{\left[1 + \left(\frac{\ln x}{b}\right)^{2a}\right]} \quad x > 1,$$

which is hazard function of GLBXII distribution.

7.4 Characterization based on certain functions of the random variable

The following proposition has already appeared in Hamedani (2013). So we will just state it here which can be used to characterize GLBXII distribution.

Proposition 7.4.1: Let $X:\Omega \to (1,\infty)$ be continuous random variable with cdf (7). Let $\psi(x)$ be differentiable function $(1,\infty)$ with $\lim_{x\to 1} \psi(x)=1$, then for $\delta \neq 1$,

$E\left(\psi(X) \middle| X > x\right) = \delta\psi(x),\ x>1,$ if and only if $\psi(X) = (1-F(x))^{\frac{1}{\delta}-1},\ x>1.$

Remark 7.4.1 It is easy to see that for certain functions, e.g. $\psi(X) = \left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-1}$ and $\delta = \frac{p}{1+p}$, Proposition 7.4.1 provides a characterization of GLBXII distribution. Clearly there are other functions $\psi(x)$, we chose the case for simplicity.

8. Maximum Likelihood Estimation

In this section, parameters estimates are derived using maximum likelihood Method. Generalized Log-Burr XII (GLBXII), Log-Burr XII (LBXII), Log-Lomax (LL), Generalized Burr XII (GBXII), Burr XII, probability distribution are fitted to real data sets of survival times of patients (Harter and Moore; 1965) and eruptions data for comparison purpose. The likelihood function for GLBXII distribution with the vector of parameters $\Phi = (a, \mathrm{b}, \mathrm{p})$ is

$$\ln L(x_i,\Phi) = n\ln 2 + n\ln \mathrm{a} + n\ln \mathrm{p} - 2na\ln \mathrm{b} - \sum_{i=1}^{n}\ln x_i + (2a-1)\sum_{i=1}^{n}\ln(\ln x_i) - (p+1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{\ln x_i}{b}\right)^{2a}\right]. \quad (57)$$

In order to compute the estimates of parameters of GLBXII distribution; the following nonlinear equations must be solved simultaneously.

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - \sum_{i=1}^{n}\ln\left[1+\left(\frac{\ln x_i}{b}\right)^{2a}\right] = 0, \quad (58)$$

$$\frac{\partial \ln L}{\partial b} = -\frac{2na}{b} + (p+1)\frac{2a}{b}\sum_{i=1}^{n}\left[\left(\frac{\ln x_i}{b}\right)^{2a}\left[1+\left(\frac{\ln x_i}{b}\right)^{2a}\right]^{-1}\right] = 0, \quad (59)$$

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} - 2n\ln \mathrm{b} + 2\sum_{i=1}^{n}\ln(\ln x_i) - 2(p+1)\sum_{i=1}^{n}\left(\frac{\ln x_i}{b}\right)^{2a}\ln\left(\frac{\ln x_i}{b}\right)\left[1+\left(\frac{\ln x_i}{b}\right)^{2a}\right]^{-1} = 0. \quad (60)$$

8.1 Survival Times Data

The GLBXII distribution is compared with LBXII, Log-Lomax, GBXII, BXII distributions. Different goodness fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov Smirnov (KS), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and likelihood ratio statistics are computed for survival times of patients (Harter and Moore; 1965) using Adequacy Model with method "L-BFGS-B" in R. Survival times of patients are 20.9, 32.2, 33.2, 39.4, 40.0, 46.8, 57.3, 58.0, 59.7, 61.1, 61.4, 54.3, 66.0, 66.3, 67.4, 68.5, 69.9, 72.4, 73.0, 73.2, 88.7, 89.3, 91.6, 93.1, 94.2, 97.7, 101.6, 101.9, 107.6, 108.0, 109.7, 110.8, 114.1, 117.5, 119.2, 120.3, 133.0, 133.8, 163.3 and 165.1.

The better fit corresponds to smaller $-2l$, W, KS, A, AIC, CAIC, BIC, and HQIC value. The maximum likelihood estimates of parameters and values of goodness of fit measures are computed via Adequacy Model with method "L-BFGS-B" in R for GLBXII distribution, its sub-models and its competing models. Table 1 displays MLEs and their standard errors (in parentheses) for data set. Table 2 displays goodness-of-fit values.

Table 1: MLEs and their standard errors (in parentheses) for data set I

Model	a	b	p	**W**	**A**
GLBXII	5.952419 (0.7519696)	7.922870 (3.0797457)	758.467637 (3410.529913)	0.04511032	0.2581986
LBXII	4.4242511 (23.5682752)		0.0772369 (0.4115119)	0.155664	1.003318
Log Lomax		575.4626 (305.9921)	17352.8960 (18353.0823)	0.09836263	0.6362552
GBXII	2.757848 (0.5950906)	296.571642 (882.8092055)	23.707684 (175.6961986)	0.04564259	0.2608821
BXII	3.07778956 (12.1986885)		0.07475319 (0.2963291)	0.1118588	0.726044

Table 2: Goodness-of-fit statistics for data set I

Model	**D(KS)**	**AIC**	**CAIC**	**BIC**	**HQIC**	$-2l$
GLBXII	0.0974	399.6035	400.2702	404.6702	401.4355	393.6036
LBXII	0.5479	579.1378	579.4621	582.5156	580.3591	575.1378
Log Lomax	0.4433	495.0194	495.3438	498.3972	496.2407	491.0194
GBXII	0.1096	399.7696	400.4363	404.8363	401.6016	393.7696
BXII	0.5251	549.1458	549.4701	552.5235	550.3671	545.1458

GLBXII distribution is best fitted than LBXII, LL, GBXII, BXII distributions because the values of all criteria are smaller for GLBXII distribution.

8.2 Waiting Times Data

The waiting times between 65 consecutive eruptions of a period (1340 hours) beginning from July 12, 1998 were noted by means of a digital watch at visitor attraction Kiama Blowhole near Sydney (Pinho et al., 2015, Silva et al., 2013, Alzaatreh et al., 2016). The waiting times data were recorded by Professor Jim Irish. The waiting times are 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35,47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9,12.

The better fit corresponds to smaller $-2l$, W, KS, A, AIC, CAIC, BIC, and HQIC value. The maximum likelihood estimates of parameters and values of goodness of fit measures are computed via Adequacy Model with method "L-BFGS-B" in R for GLBXII distribution, its sub-models and its competing models. Table 3 displays MLEs and their standard errors (in parentheses) for data set II. Table 4 displays goodness-of-fit values.

Table 3: MLEs and their standard errors (in parentheses) for data set II

Model	a	b	p	W	A
GLBXII	2.249308 (0.2244228)	16.271831 (16.5155409)	809.148977 (3604.582513)	0.1014155	0.7435557
LBXII	6.76354212 (24.4391332)		0.06296473 (0.2275536)	0.2170178	1.442308
Log Lomax		388.6678 (119.5571)	12683.5477 (7689.4699)	0.112869	0.8263072
GBXII	1.583643 (0.3563377)	60.724678 (63.0856066)	2.407626 (2.7439891)	0.1367699	0.9573555
BXII	4.78229724 (17.0194677)		0.06251006 (0.2225039)	0.1288313	0.9263526

Table 4: Goodness-of-fit statistics for data set II

Model	**D(KS)**	**AIC**	**CAIC**	**BIC**	**HQIC**	$-2l$
GLBXII	0.1022	593.0598	593.4598	599.5364	595.6112	587.0598
LBXII	0.4483	731.0607	731.2574	735.3785	732.7617	727.0606
Log Lomax	0.2888	638.1969	638.3936	642.5147	639.8979	634.1968
GBXII	0.1112	598.3175	598.7175	604.7941	600.8689	592.3174
BXII	0.4473	714.7839	714.9806	719.1017	716.4849	710.7838

GLBXII distribution is best fitted than LBXII, LL, GBXII, BXII distribution because the values of all criteria are smaller for GLBXII distribution.

9. Concluding Remarks

We have developed GLBXII distribution on the basis of the GLPE. We have studied certain properties including structural properties, plots, sub-models, moments, factorial moments, moments of order statistics, L-moments, TL-moments, incomplete moments, inequality measures, residual life functions, reliability and uncertainty measures and compounding. The GLBXII distribution is characterized via different techniques. Maximum Likelihood estimates are computed. Goodness of fit shows that GLBXII distribution is better fit. Two applications of the GLBXII model to survival times of patients and eruptions data are illustrated to show significance and flexibility of GLBXII distribution. We have proved that GLBXII distribution is empirically better for lifetime applications.

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Appendix A

Theorem 1: Let (Ω, F, P) be given probability space and let $H = [a_1, a_2]$ an interval with $a_1 < a_2$
$(a_1 = -\infty, a_2 = \infty)$. Let $X : \Omega \rightarrow [a_1, a_2]$ be a continuous random variable with distribution function F and Let q be real function defined on $H = [a_1, a_2]$ such that $E[q(X) | X \geq x] = \eta(x)$ $x \in H$ is defined with some real function $\eta(x)$ should be in simple form. Assume that $q(x) \varepsilon C([a_1, a_2]), \eta(x) \varepsilon C^2([a_1, a_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[a_1, a_2]$: To conclude, assume that the equation $q(x) = \eta(x)$ has no real solution in the inside of $[a_1, a_2]$.Then F is obtained from the functions $q(x)$ *and* $\eta(x)$ as $F(x) = \int_a^x k \left| \frac{\eta'(t)}{\eta(t) - q(t)} \right| \exp(-s(t)) dt$, where $s(t)$ is the solution of equation $s'(t) = \frac{\eta'(t)}{\eta(t) - q(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF = 1$.