Parameter Estimations and Optimal Design of Simple Step-Stress Model for Gamma Dual Weibull Distribution

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Abstract

This paper considers life-testing experiments and how it is effected by stress factors: namely temperature, electricity loads, cycling rate and pressure. A major type of accelerated life tests is a step-stress model that allows the experimenter to increase stress levels more than normal use during the experiment to see the failure items. The test items are assumed to follow Gamma Dual Weibull distribution. Different methods for estimating the parameters are discussed. These include Maximum Likelihood Estimations and Confidence Interval Estimations which is based on asymptotic normality generate narrow intervals to the unknown distribution parameters with high probability. MathCAD (2001) program is used to illustrate the optimal time procedure through numerical examples.

Keywords: Accelerate Life Testing; Maximum Likelihood Estimator; Information Matrix; Confidence Interval Estimators; Dual Weibull Distributions.

Introduction

The Accelerated Life Tests ALTs play pivotal role in reliability analysis of products or systems. During an experiment, the stress levels increase under normal operating conditions and at pre-fixed time points on test units to obtain quickly information on the parameters of the lifetime distributions in a short time. In addition, failure resulting from units of the experiment is probably due to several stress factors, including temperature, cycling, pressure, vibration and voltage, etc. This procedure is called a Step-Stress test SSt.

Many authors mentioned different choices for the SSt. (Drop et al 1996) provided a Bayes estimation for step-stress model and they assumed the failure times have exponential distribution while the specification of strict compliance to the transformation function of time is not desired. (Dharmadhikari & Monsur 2003) assumed that the scale parameter of Weibull and Lognormal models depend on the age of the present levels. They provided a parametric model to the failure time distribution for the SSt and estimated the parameters involved in it. (Chen et al 2006) considered the exponential population as a failure time to SSt. They applied the tampered failure rate model to obtain some measures such evidence for the existence, strong consistency, uniqueness and the asymptotic normality of the MLE of mean. Under a cumulative exposure model, (Al-Ghamdi & Hassan 2009) considered the problem using the Lomax distribution. By minimizing with respect to the asymptotic variance of the MLE of a given 50th percentile and the change time scale parameter, they obtained the optimum test plan.

(Donghoon, H. & Balakrishnan2010) considered the simple SSt under time constant when the failure times of the different risk factors are distributed as exponential distribution. Under the assumption of a cumulative exposure model, they derived the

MLEs of the unknown parameters. (Abd-Elfattah and Al-Harbey2010) studied the estimation problem when the lifetime distribution of the test items is following Burr type III distribution. They derived the MLEs for the acceleration factor and distribution parameters in case type II censored samples. Based on type II right censored data from the SSt, (Lee et al 2013) studied the problem for exponential products. They performed tests of hypotheses about model parameters based on the likelihood method.

New family of distributions namely ΓG due to (Zagrafos and Balakrishnan 2009). They added a shape parameter to the baseline G distribution. In (2015), (Castellares and Lemonte) studied the concept of Zagrafos and Balakrishnan and they proposed a very useful explanation for its pdf. Another family of distributions was introduced by (Ristic and Balakrishnan2012) based on random variables taken from gamma distribution. The new family say $\Gamma 2.\mathrm{G}$ may be regarded to ΓG family.

(Castellares and Lemonte 2016) introduced three interesting propositions to study the relation between ΓG and $\Gamma 2.\mathrm{G}$ families and used the last family to generate the new three parameters gamma dual Weibull $\Gamma 2.\mathrm{W}$ distribution which has CDF and pdf respectively as:

$$F_X(x) = 1 - \frac{\gamma\left(\theta, -\log\left(1 - e^{-\alpha x^{\beta}}\right)\right)}{\Gamma(\theta)} \qquad ; x > 0 \tag{1}$$

and

$$f_X(x) = \frac{\alpha\beta}{\Gamma(\theta)} x^{\beta-1} e^{-\alpha x^{\beta}} \left[-\log\left(1 - e^{-\alpha x^{\beta}}\right)\right]^{\theta-1} \qquad ; x > 0 \tag{2}$$

Where $\gamma(\delta, t) = \int_0^t w^{\delta-1} e^{-w}\, dw$ is the incomplete gamma function, $\theta, \beta > 0$ are the shape parameters and $\alpha > 0$ is the scale parameter.

Note that, if $\theta = 1$ in (2), then the pdf of the Weibull distribution comes. If $\beta = 1$, then (2) represents the pdf of the gamma dual exponential distribution. When $\beta = 2$, the gamma dual Rayleigh distribution arises. And if $\theta = \beta = 1$, the exponential distribution holds.

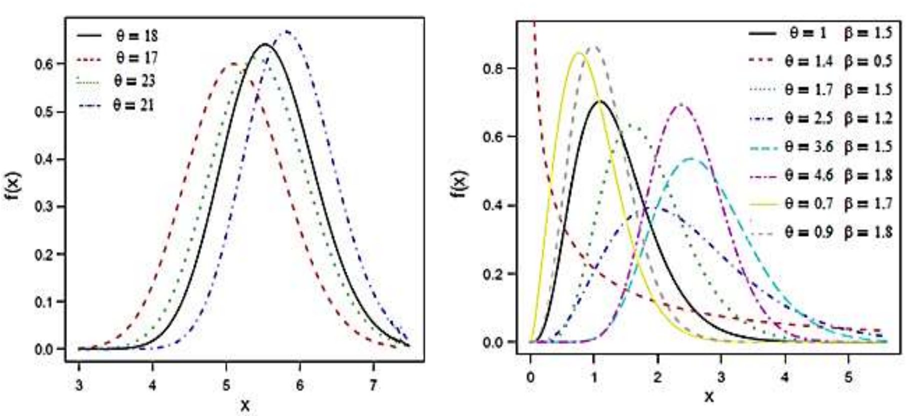

Fig. 1. Different forms for pdf of the $\Gamma 2.\mathrm{W}$
with various values of the parameters (θ, β) at $\alpha = 1$

This paper consists of four sections corresponding to 2, 3, 4 and 5, respectively. section 2 discusses the model description and assumptions. section 3, provides the likelihood function estimators and the asymptotic Fisher information matrix which contains the second partial derivatives of logarithm of likelihood function. Confidence intervals for the parameters are examined based on asymptotic normality in section 4. The last section considers a numerical investigation to study the properties of the new estimators.

1. Model Description and Assumptions

Based on the CDF of the $\Gamma 2.\mathrm{W}$ distribution which in (4), it contains on failure time of test item in case constant stress during the test intervals. And the assumptions are made as below:

i. There is a relation that represents function between stress level Z_i and scale parameter α_i as $\log(\alpha_i) = \lambda_0 + \lambda_1 Z_i$ where λ_0, λ_1 are unknown parameters depend on the way and outcomes of the test and $i = 1, 2$.

ii. The entire n_i, $i = 1, 2$ items are put at the first stress level Z_1, at the beginning, continuing until time τ. Next, the stress is varied to a high level Z_2, and the test ongoing until all items failed.

iii. At the time x_{ij}, $j = 1, 2, \ldots, n_i$ under a stress level Z_i, $i = 1, 2$, all of n_i failure are observed.

iv. In SSt which is the one of ALTs, the cumulative exposure model is:

$$F(x) = \begin{cases} F_1(x) & 0 < x < \tau \\ F_2(x - \tau + \tau') & \tau < x < \infty \end{cases} \quad (3)$$

where $\tau' = \tau\left(\frac{\alpha_2}{\alpha_1}\right)$ which is solution of $F_1(\tau) = F_2(\tau')$ for τ'. So, the cumulative exposure model for gamma dual Weibull distribution is obtained as follow:

$$F(x) = \begin{cases} F_1(x) = 1 - \dfrac{\gamma\left(\theta, -\log\left(1 - e^{-\alpha_1 x^{\beta}}\right)\right)}{\Gamma(\theta)} & ; 0 \le x < \tau \\ F_2(x) = 1 - \dfrac{\gamma\left(\theta, -\log\left(1 - e^{-\alpha_2\left(x - \tau\left(1 - \frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}\right)\right)}{\Gamma(\theta)} & ; \tau \le x < \infty \end{cases} \quad (4)$$

and the pdf of the failure time is:

$$f(x)=\begin{cases} f_1(x)=\dfrac{\alpha_1\beta}{\Gamma(\theta)}x^{\beta-1}e^{-\alpha_1 x^{\beta}}\left[-\ln\left(1-e^{-\alpha_1 x^{\beta}}\right)\right]^{\theta-1} & ;0\le x<\tau \\ f_2(x)=\dfrac{\alpha_2\beta}{\Gamma(\theta)}\left(x-\tau\left(1-\dfrac{\alpha_2}{\alpha_1}\right)\right)^{\beta-1} & \\ \quad\times e^{-\alpha_2\left(x-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}\left[-\ln\left(1-e^{-\alpha_2\left(x-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}\right)\right]^{\theta-1} & ;\tau\le x<\infty \end{cases} \quad (5)$$

In next section, the MLEs and asymptotic Fisher information matrix are provided.

Likelihood Function and Asymptotic Fisher Information Matrix

Let the failure times $x_{ij},\ j=1,2,\ldots,n_i\ ,\ i=1,2$ and $n_1,\ n_2$ are the number of items which are fail at the first stress level Z_1 and high stress Z_2 respectively. Then, their likelihood and log likelihood functions respectively are:

$$L(x;\theta,\beta,\alpha_1,\alpha_2)=\left(\frac{\alpha_1\beta}{\Gamma(\theta)}\right)^{n_1}\left(\frac{\alpha_2\beta}{\Gamma(\theta)}\right)^{n_2}\times\prod_{j=1}^{n_1}x_i^{\beta-1}e^{-\alpha_1 x_i^{\beta}}\left[-\ln\left(1-e^{-\alpha_1 x_i^{\beta}}\right)\right]^{\theta-1}$$

$$\times\prod_{j=1}^{n_2}\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta-1}e^{-\alpha_2\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}$$

$$\times\left[-\ln\left(1-e^{-\alpha_2\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}\right)\right]^{\theta-1} \quad . \quad (6)$$

and

$$\begin{aligned} l(x;\theta,\beta,\alpha_1,\alpha_2)&=n_1\log(\alpha_1)+n_2\log(\alpha_2)+(n_1+n_2)\ln(\beta) \\ &-(n_1+n_2)\ln(\Gamma(\theta))+(\beta-1) \\ &\times\left(\sum_{i=1}^{n_1}\ln(x_i)+\sum_{i=1}^{n_2}\ln\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)\right)-\alpha_1\sum_{i=1}^{n_1}x_i^{\beta} \\ &-\alpha_2\sum_{i=1}^{n_2}\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}+(\theta-1)\left[\sum_{i=1}^{n_1}\ln\left(-\ln\left(1-e^{-\alpha_1 x_i^{\beta}}\right)\right)\right. \\ &\left.+\sum_{i=1}^{n_2}\ln\left(-\ln\left(1-e^{-\alpha_2\left(x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right)^{\beta}}\right)\right)\right] \end{aligned} \quad . \quad (7)$$

The first derivatives of $l(x;\theta,\beta,\alpha_1,\alpha_2)$ with respect to θ,β,α_1 and α_2 are, respectively:

$$\frac{\partial l}{\partial\theta}=-(n_1+n_2)\psi(\theta)+\sum_{i=1}^{n_1}\ln(-\ln(\nu))+\sum_{i=1}^{n_2}\ln(-\ln(c)) \quad . \quad (8)$$

$$\frac{\partial l}{\partial \beta}=\frac{n_1+n_2}{\beta}+\sum_{i=1}^{n_1}\ln(x_i)+\sum_{i=1}^{n_2}\ln(w)-\alpha_1.\sum_{i=1}^{n_1}x_i^{\beta}\ln(x_i)$$
$$-\alpha_2.\sum_{i=1}^{n_2}w^{\beta}\ln(w)+(\theta-1)\Bigg[\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}.\ln(x_i)\frac{1-\nu}{(\nu).\ln(\nu)}$$
$$+\alpha_2.\sum_{i=1}^{n_2}w^{\beta}.\ln(w).\frac{1-c}{(c)\ln(c)}\Bigg] \qquad (9)$$

$$\frac{\partial l}{\partial \alpha_1}=\frac{n_1}{\alpha_1}-(\beta-1)\sum_{i=1}^{n_2}\left(\frac{\tau\alpha_2}{\alpha_1^2.w}\right)+\alpha_2\sum_{i=1}^{n_2}\left(\frac{\beta\tau\alpha_2}{\alpha_1^2.w}\right)+(\theta-1)\Bigg[\sum_{i=1}^{n_1}x_i^{\beta}\frac{(1-\nu)}{(\nu).\ln(\nu)}$$
$$-\left(\frac{\alpha_2}{\alpha_1}\right)^2\beta\tau\sum_{i=1}^{n_2}w^{\beta-1}\frac{1-c}{c\ln(c)}\Bigg] \qquad (10)$$

$$\frac{\partial l}{\partial \alpha_2}=\frac{n_2}{\alpha_2}+(\beta-1)\sum_{i=1}^{n_2}\left(\frac{\tau}{\alpha_1 w}\right)-\sum_{i=1}^{n_2}w^{\beta}\left(1+\beta\tau\frac{\alpha_2}{\alpha_1 w}\right)-(\theta-1)$$
$$\times\sum_{i=1}^{n_2}\frac{1-c}{c\ln(c)}w^{\beta-1}\left[w+\beta\tau\frac{\alpha_2}{\alpha_1}\right] \qquad (11)$$

where $w=\left[x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right]$, $\psi(\theta)=\frac{d}{d\theta}\ln\left[\Gamma(\theta)\right]=\frac{\Gamma'(\theta)}{\Gamma(\theta)}$ is the digamma function .

$\nu=1-e^{-\alpha_1 x_i^{\beta}}$ and $c=1-e^{-\alpha_2 w^{\beta}}$.

Obviously, there is no closed form for θ,β,α_1 and α_2 in (8), (9), (10) and (11).So, Newton–Raphson method is used to solve these equations in numerical analysis. It is an iterative method for solving equations $f(b)=0$ where $f(b)$ is assumed to have a continuous derivative $f'(b)$. Given a function $f(b)$ and its derivative $f'(b)$, a first guess b_0 is initialed . Then, an approximation of b_1 is $b_0-\frac{f(b_0)}{f'(b_0)}$ and an approximation of b_2 is $b_1-\frac{f(b_1)}{f'(b_1)}$ and so on for number of iterations r or if $|b_{r+1}-b_r|\le\xi$ where b_r is the r^{th} estimate. (see Kotz et al (2003) [11].

Now, the second derivatives of $l(x;\theta,\beta,\alpha_1,\alpha_2)$ with respect to θ,β,α_1 and α_2 are obtained to construct the asymptotic Fisher information matrix as follows:

$$I=-E\begin{bmatrix}\frac{\partial^2 l}{\partial\theta^2} & & & \\ \frac{\partial^2 l}{\partial\beta\,\partial\theta} & \frac{\partial^2 l}{\partial\beta^2} & & \\ \frac{\partial^2 l}{\partial\alpha_1\,\partial\theta} & \frac{\partial^2 l}{\partial\alpha_1\,\partial\beta} & \frac{\partial^2 l}{\partial\alpha_1^2} & \\ \frac{\partial^2 l}{\partial\alpha_2\,\partial\theta} & \frac{\partial^2 l}{\partial\alpha_2\,\partial\beta} & \frac{\partial^2 l}{\partial\alpha_2\,\partial\alpha_1} & \frac{\partial^2 l}{\partial\alpha_2^2}\end{bmatrix}.$$

where

$$\frac{\partial^2 l}{\partial\theta^2}=-(n_1+n_2)\left[\frac{\Gamma(\theta)\Gamma''(\theta)-\left[\Gamma'(\theta)\right]^2}{\left[\Gamma(\theta)\right]^2}\right]. \tag{12}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta^2}=&-\frac{(n_1+n_2)}{\beta^2}-\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}.\left[\ln(x_i)\right]^2-\alpha_2\sum_{i=1}^{n_2}w^{\beta}.\left[\ln(w)\right]^2\\&+(\theta-1)\Bigg\{\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}.\left[\ln(x_i)\right]^2\frac{1-\nu}{\nu\cdot\ln(\nu)}\\&\times\left[1-\alpha_1x_i^{\beta}-\alpha_1x_i^{\beta}\frac{(1-\nu)}{\nu}-\alpha_1x_i^{\beta}\frac{1-\nu}{\nu\cdot\ln(\nu)}\right]\\&+\alpha_2\sum_{i=1}^{n_2}w^{\beta}.\left[\ln(w)\right]^2\frac{1-c}{c\cdot\ln(c)}\\&\times\left[1-\alpha_2w^{\beta}-\alpha_2w^{\beta}\frac{1-c}{c}-\alpha_2w^{\beta}\frac{1-c}{c\cdot\ln(c)}\right]\Bigg\}\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha_1^2}=&-\frac{n_1}{\alpha_1^2}-(\beta-1)\sum_{i=1}^{n_2}\frac{2\tau}{w}\left(\frac{\alpha_2}{\alpha_1}\right)^2-\left(\frac{\tau}{w}\right)^2\left(\frac{\alpha_2^2}{\alpha_1^4}\right)\alpha_2\sum_{i=1}^{n_2}w^{\beta}.\left[\ln(w)\right]^2\\&-\alpha_2\sum_{i=1}^{n_2}\left[w^{\beta-2}\beta^2\tau^2\left(\frac{\alpha_2^2}{\alpha_1^4}\right)+2w^{\beta-1}\beta\tau\left(\frac{\alpha_2}{\alpha_1^3}\right)-w^{\beta-2}\beta\tau^2\left(\frac{\alpha_2^2}{\alpha_1^4}\right)\right]\\&-(\theta-1)\Bigg\{\sum_{i=1}^{n_1}\left[\left(x_i^{\beta}\right)^2\left[\frac{1-\nu}{\nu\cdot\ln(\nu)}-\frac{(1-\nu)^2}{\nu^2\cdot\ln(\nu)}-\frac{(1-\nu)^2}{\nu^2\cdot(\ln(\nu))^2}\right]\right]\\&-\sum_{i=1}^{n_2}\left[\alpha_2^3w^{\beta-1}\left[(\beta-1)\frac{1-\eta}{\eta\cdot\ln(\eta)}\left(\frac{\tau^2}{\alpha_1^4}\right)-2\beta\left(\frac{\tau}{\alpha_1^3}\right)\right.\right.\\&\left.\left.-\beta^2\left(\frac{\tau^2}{\alpha_1^4}\right)-\beta^2\left(\frac{\tau^2}{\alpha_1^4\eta}\right)-\beta^2\left(\frac{\tau^2}{\alpha_1^4\eta\ln(\eta)}\right)\right]\right]\Bigg\}\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha_2^2}=&-\frac{(n_1+n_2)}{\beta}-\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}.\left[\ln(x_i)\right]^2-\alpha_2\sum_{i=1}^{n_2}w^{\beta}.\left[\ln(w)\right]^2\\&-(\theta-1)\Bigg\{\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}.\left[\ln(x_i)\right]^2\frac{1-v}{v\cdot\ln(v)}\\&\times\left[1-\alpha_1x_i^{\beta}-\alpha_1x_i^{\beta}\frac{1-v}{v}-\alpha_1x_i^{\beta}\frac{1-v}{v\cdot\ln(v)}\right]\\&-\alpha_2\sum_{i=1}^{n_2}w^{\beta}.\left[\ln(w)\right]^2\frac{1-c}{w\cdot\ln(w)}\\&\times\left[1-\alpha_2w-\alpha_2w\frac{1-c}{w}-\alpha_2w\frac{1-c}{w\cdot\ln(w)}\right]\Bigg\}\end{aligned} \tag{15}$$

$$\frac{\partial^2 l}{\partial\theta\,\partial\beta}=\frac{\partial^2 l}{\partial\beta\,\partial\theta}=\alpha_1\sum_{i=1}^{n_1}x_i^{\beta}\,\ln(x_i)\frac{1-\nu}{\nu\cdot\ln(\nu)}+\alpha_2\sum_{i=1}^{n_2}\left(w^{\beta}\ln(w)\frac{1-c}{c\ln(c)}\right). \tag{16}$$

$$\frac{\partial^2 l}{\partial\theta\,\partial\alpha_1}=\frac{\partial^2 l}{\partial\alpha_1\,\partial\theta}=\sum_{i=1}^{n_1}x_i^{\beta}\;\frac{1-\nu}{\nu\cdot\ln(\nu)}+\left(\frac{\alpha_2}{\alpha_1}\right)^2\beta\tau\sum_{i=1}^{n_2}w^{\beta-1}\frac{1-c}{c\ln(c)}. \tag{17}$$

$$\frac{\partial^2 l}{\partial\theta\,\partial\alpha_2}=\frac{\partial^2 l}{\partial\alpha_2\,\partial\theta}=-\sum_{i=1}^{n_2}\frac{1-c}{c\ln(c)}\left(w^{\beta}-\frac{\alpha_2\beta\tau}{\alpha_1}w^{\beta-1}\right). \tag{18}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta\,\partial\alpha_1}=\frac{\partial^2 l}{\partial\alpha_1\,\partial\beta}&=-n_2\left(\frac{\alpha_2}{\alpha_1}\right)\frac{\tau}{w}-\sum_{i=1}^{n_1}x_i^{\beta}\ln(x_i)+\left(\frac{\alpha_2}{\alpha_1}\right)^2\sum_{i=1}^{n_2}w^{\beta}\tau(\beta+1)\\&+(\theta-1)\left\{\sum_{i=1}^{n_1}\alpha_1x_i^{\beta}\ln(x_i)\left(\frac{1-\nu}{\nu}\right)\left[\frac{1}{\alpha_1}-x_i^{\beta}\left(1-\frac{1-\nu}{\nu}-\frac{1-\nu}{\nu\ln(\nu)}\right)\right]\right.\\&-\left(\frac{\alpha_2}{\alpha_1}\right)^2\left[\sum_{i=1}^{n_2}w^{\beta-1}\ln(w)\beta\tau\frac{1-c}{c\ln(c)}\right]\\&\left.\times\left[1+\frac{1}{\ln(w)}-\alpha_2w^{\beta}\left(1+\frac{1-c}{c}+\frac{1-c}{c\ln(c)}\right)\right]\right\}.\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta\,\partial\alpha_2}=\frac{\partial^2 l}{\partial\alpha_2\,\partial\beta}&=\sum_{i=1}^{n_2}\frac{\tau}{\alpha_1w}-\sum_{i=1}^{n_2}w^{\beta-1}\ln(w)\frac{\alpha_2}{\alpha_1}\beta\tau\left[1+\frac{1}{\ln(w)}\right]\\&-(\theta-1)\frac{d}{d\alpha_2}\left[\alpha_2.\sum_{i=1}^{n_2}w^{\beta}.\ln(w).\frac{1-c}{c\ln(c)}\right]\end{aligned}. \tag{20}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha_1\,\partial\alpha_2}=\frac{\partial^2 l}{\partial\alpha_2\,\partial\alpha_1}&=(\beta-1)\sum_{i=1}^{n_2}\frac{\tau}{\alpha_1^2w}\left(\frac{\tau\alpha_2}{\alpha_1w}-1\right)+\sum_{i=1}^{n_2}w^{\beta}\beta\tau\frac{\alpha_2}{\alpha_1^2w}\\&-\alpha_2\sum_{i=1}^{n_2}w^{\beta}\beta\left[\frac{\alpha_2\tau}{\alpha_1^3}w^2-\frac{\tau}{\alpha_1^2}w-\frac{\alpha_2\beta\tau^2}{\alpha_1^3}w^2\right]\\&-(\theta-1)\frac{d}{d\alpha_2}\left[\sum_{i=1}^{n_2}\left(\frac{\alpha_2}{\alpha_1}\right)^2w^{\beta-1}\beta\tau\left(\frac{1-c}{c\ln(c)}\right)\right]\end{aligned}. \tag{21}$$

where $\nu=1-e^{-\alpha_1x_i^{\beta}}$, $w=\left[x_i-\tau\left(1-\frac{\alpha_2}{\alpha_1}\right)\right]$, $\psi(\theta)=\frac{d}{d\theta}\ln\left[\Gamma(\theta)\right]=\frac{\Gamma'(\theta)}{\Gamma(\theta)}$ is the digamma function and $c=\frac{e^{-\alpha_2w^{\beta}}}{\left(1-e^{-\alpha_2w^{\beta}}\right)\ln\left(1-e^{-\alpha_2w^{\beta}}\right)}$.

Approximate Confidence Interval

Let $X_1,X_2,\ldots,X_n$ be two independent random samples drawn from $\Gamma 2.\mathrm{W}$ distribution as Eq. (5) with parameters θ,β,α_1 and α_2. These two random samples are used at different stress values of Z_1, Z_2. Then, the maximum likelihood estimation $\hat{\Omega}$ of Ω is asymptotically normally distributed with mean $\hat{\Omega}$ and variance $\sigma^2(\hat{\Omega})$ under regularity conditions and large sample. It can be constructed as:

$$P\left[\hat{\Omega}-z_{\omega/2}.\sigma\left(\hat{\Omega}\right)\leq\Omega\leq\hat{\Omega}+z_{\omega/2}.\sigma\left(\hat{\Omega}\right)\right]=1-\gamma .$$

Now, using following equations and confidence level $1-\omega=0.95$, the lower and upper confidence intervals can be constructed as:

$$\begin{aligned}
&\hat{\theta}_0-z_{\omega/2}.\sigma\left(\hat{\theta}_0\right)\leq\theta\leq\hat{\theta}_0+z_{\omega/2}.\sigma\left(\hat{\theta}_0\right),\\
&\hat{\beta}_0-z_{\omega/2}.\sigma\left(\hat{\beta}_0\right)\leq\beta\leq\hat{\beta}_0+z_{\omega/2}.\sigma\left(\hat{\beta}_0\right),\\
&\hat{\alpha}_{10}-z_{\omega/2}.\sigma\left(\hat{\alpha}_{10}\right)\leq\alpha_1\leq\hat{\alpha}_{10}+z_{\omega/2}.\sigma\left(\hat{\alpha}_{10}\right),\\
&\hat{\alpha}_{20}-z_{\omega/2}.\sigma\left(\hat{\alpha}_{20}\right)\leq\alpha_2\leq\hat{\alpha}_{20}+z_{\omega/2}.\sigma\left(\hat{\alpha}_{20}\right).
\end{aligned} \tag{22}$$

Simulation Study and Data Analysis

The computer programs MathCAD (2001) is used to obtain numerical illustration for the last theoretical results. 1000 samples generated from gamma dual Weibull distribution with parameters (θ,β,α_1) and (θ,β,α_2) are used, respectively at different values of Z_1, Z_2 with various size samples $n=n_2,+\ n_2$ in table 1, 2 and 3.

Relative bias (R Bias) $RE=\left|\frac{\hat{\alpha}-\alpha}{\alpha}\right|$, mean square error (MSE) which is $MSE(\hat{\alpha})=E\left[\left(\hat{\alpha}-\alpha\right)^2\right]$, and relative error (RE) $RE(\hat{\theta})=\frac{\sqrt{MSE(\hat{\alpha})}}{\alpha}$ are calculated. Also, the lower and upper confidence intervals of the acceleration factor and both two shape and scale parameters are obtained. The steps of Simulation are:

1. The random samples are generate from cumulative exposure for gamma dual Weibull distribution as Eq. (2) at various values of Z_1, Z_2 with various size samples $n=n_2\ +\ n_2$. Then, calculate $\log(\alpha_i)=\lambda_0+\lambda_1 Z_i$; $i=1,2$.
2. For each sample, the non-linear likelihood system (3.3), (3.4) and (3.5) are solved for $(\theta,\beta\ ,\lambda_1,\ \lambda_2)$ using Newton–Raphson method which was illustrated.
3. The (R Bias), (MSE) and (RE) of the four parameter estimators are obtained. Also, for different sample sizes, asymptotic variance and covariance matrix of the estimators are calculated.
4. The lower and upper confidence intervals at significant level $\omega=0.05$ of the two shape and scale parameters are construct.

Now, from table (1) and (2), note that, the (R Bias), (MSE) and (RE) of the four parameters $(\theta, \beta, \lambda_1, \lambda_2)$ are decreasing by increasing the sample size n. Also, in table (3), the lower and upper confidence intervals of the acceleration factor and two shape and scale parameters are asymptotic, so the length of interval is decreasing by increasing the sample size n.

Table 1: R BIAS, MSE and RE Values for parameters $(\theta, \beta, \lambda_1, \lambda_2)$ at different stress levels and different sample sizes

$n_1 + n_2$	$\theta, \beta, \alpha_1, \alpha_1$	$Z_0 = 0.3,\ Z_1 = 0.7,$			$Z_0 = 0.25,\ Z_1 = 1,$		
		R BIAS	MSE	RE	R BAIS	MSE	RE
50	θ	0.0708	0.1988	0.0788	0.0478	0.1348	0.0638
	β	0.7558	0.4278	0.6508	0.9478	0.7018	0.8328
	α_1	0.2428	0.1998	0.2438	0.2998	0.3808	0.3008
	α_2	0.2664	0.2048	0.2541	0.3007	0.3976	0.3312
60	θ	0.0228	0.0598	0.0408	0.0378	0.1008	0.0538
	β	0.5918	0.3188	0.5628	0.3418	0.6798	0.8188
	α_1	0.2158	0.1698	0.2178	0.2978	0.3798	0.3048
	α_2	0.2318	0.1766	0.2248	0.2994	0.3519	0.3117
70	θ	0.0148	0.0308	0.0288	0.0248	0.0428	0.0338
	β	0.5908	0.2818	0.5298	0.2868	0.5328	0.7268
	α_1	0.1998	0.1328	0.2008	0.2898	0.3558	0.2908
	α_2	0.2167	0.1634	0.2413	0.2988	0.3742	0.3107
80	θ	0.0088	0.0298	0.0208	0.0118	0.0388	0.0318
	β	0.5458	0.2428	0.4928	0.2648	0.5018	0.7048
	α_1	0.1748	0.1418	0.1778	0.2768	0.3248	0.2778
	α_2	0.1883	0.1573	0.1982	0.2944	0.3370	0.2811
90	θ	0.0028	0.0148	0.0198	0.0018	0.0188	0.0228
	β	0.4188	0.2328	0.4828	0.1558	0.4288	0.6518
	α_1	0.1568	0.1128	0.1638	0.2458	0.2498	0.2468
	α_2	0.1733	0.1289	0.1801	0.2539	0.2644	0.2704
100	θ	-0.0082	0.0148	0.0368	-0.0012	0.0128	0.0188
	β	0.3298	0.1478	0.3878	0.1418	0.0958	0.3178
	α_1	0.1488	0.0958	0.1518	0.2428	0.2458	0.2438
	α_2	0.1577	0.1148	0.1631	0.2840	0.2543	0.2644

Table 2: R BIAS, MSE and RE Values for parameters $(\theta, \beta, \lambda_1, \lambda_2)$ at different stress levels and different sample sizes

$n_1 + n_2$	$\theta, \beta, \alpha_1, \alpha_1$	$Z_0 = 0.4,\ Z_1 = 1,$			$Z_0 = 0.8,\ Z_1 = 1,$		
		R BIAS	MSE	RE	R BAIS	MSE	RE
50	θ	0.08176	0.24276	0.08776	0.11476	0.79476	0.16376
	β	0.48276	0.83776	0.90876	0.64876	0.64676	0.79876
	α_1	0.34376	0.49876	0.34476	0.42476	0.75676	0.42576
	α_2	0.36112	0.51830	0.36733	0.50183	0.84117	0.48347
60	θ	0.05176	0.12576	0.06176	0.10476	0.66576	0.15176
	β	0.40376	0.66976	0.81276	0.50576	0.53276	0.72576
	α_1	0.33476	0.47476	0.33576	0.41276	0.71276	0.41276
	α_2	0.39971	0.53280	0.45921	0.57836	0.87631	0.54790
70	θ	0.04176	0.09376	0.05176	0.11676	0.45576	0.11376
	β	0.36276	0.66476	0.80976	0.41176	0.41476	0.64076
	α_1	0.32376	0.44276	0.32476	0.40676	0.69276	0.40676
	α_2	0.33581	0.48972	0.38412	0.55764	0.79431	0.40994
80	θ	0.02676	0.04876	0.03276	0.06176	0.38676	0.06976
	β	0.16576	0.42476	0.64876	0.35576	0.28176	0.53076
	α_1	0.26176	0.36676	0.29476	0.40176	0.63676	0.38976
	α_2	0.37581	0.39428	0.36117	0.51336	0.69480	0.41881
90	θ	0.02576	0.04076	0.02376	0.05276	0.16576	0.06376
	β	0.16276	0.37376	0.60876	0.23476	0.24376	0.49376
	α_1	0.29276	0.36376	0.29376	0.38976	0.63376	0.38876
	α_2	0.34781	0.39722	0.34337	0.41127	0.66279	0.40059
100	θ	0.01276	0.02176	0.03676	0.03376	0.13576	0.06476
	β	0.06876	0.29976	0.54576	0.16076	0.20476	0.20176
	α_1	0.28976	0.29076	0.26276	0.38176	0.61576	0.38376
	α_2	0.31448	0.34655	0.29710	0.39979	0.67425	0.39576

Table 3: Upper and lower bounds of Confidence intervals at significant level 0.05

n_1+n_2	$\theta,\beta,\alpha_1,\alpha_1$	$Z_0=0.4,\ Z_1=1,$			$Z_0=0.8,\ Z_1=1,$		
		LL	UL	Length	LL	UL	Length
50	θ	4.75346	5.47046	0.50346	3.90846	5.38446	1.26246
	β	-0.00154	0.88446	0.67246	-0.10354	0.98346	0.87346
	α_1	1.32146	2.02846	0.49346	0.63246	1.83646	0.99046
	α_2	1.95220	2.20461	0.54831	0.72490	2.00431	1.45661
60	θ	4.90946	5.61546	0.49246	5.43546	6.43246	0.78346
	β	-0.02354	0.84346	0.65346	0.14046	0.86346	0.50946
	α_1	1.19846	1.87946	0.46746	0.75846	1.81146	0.83946
	α_2	1.31174	1.95482	0.85439	0.94721	1.94726	0.99438
70	θ	4.71646	5.40746	0.47746	4.93446	5.82046	0.67246
	β	-0.05654	0.79546	0.63846	0.27446	0.95946	0.47146
	α_1	1.17446	1.84546	0.45746	0.80546	1.85646	0.83746
	α_2	1.34720	1.97354	0.64812	0.97724	1.74590	0.95661
80	θ	4.69046	5.27846	0.37446	5.02846	5.84846	0.61146
	β	0.68846	0.92646	0.02446	0.17746	0.83046	0.43946
	α_1	1.19346	1.85046	0.44346	0.92446	1.84146	0.70346
	α_2	1.34781	1.94570	0.55899	0.97458	1.90047	0.97248
90	θ	4.73846	5.22646	0.27446	4.71846	5.51246	0.53546
	β	0.69446	0.92946	0.02146	0.24946	0.88446	0.42146
	α_1	1.34546	1.87946	0.32046	1.09646	1.79746	0.48746
	α_2	1.45733	1.95670	0.84572	1.24910	1.97354	0.65712
100	θ	4.58646	5.16746	0.36746	4.80746	5.51646	0.49546
	β	0.75446	0.96646	0.00154	0.31846	0.91046	0.37846
	α_1	1.36046	1.89046	0.31646	1.16046	1.85846	0.48446
	α_2	1.66482	2.0045	0.51034	1.30461	1.94264	0.61273

Conclusion

This paper studied a step stress which is a major type of accelerate life tests by assuming that data follow Gamma Dual Weibull distribution. Different methods for estimating the parameters are discussed including Maximum Likelihood Estimations, Fisher Information Matrix and Confidence Interval Estimations. In simulation study, three criteria measures have been calculated for the four parameters $(\theta,\beta,\lambda_1,\lambda_2)$. These measures are decreasing by increasing the sample size n. Also, the lower and upper confidence intervals of the acceleration factor and two shape and scale parameters are obtained. The length of interval is decreasing by increasing the sample size n .

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