Moments of Random Variables based on Bernoulli Trials

Ahmed Zogo Memon
National College of Business Administration and Economics
Lahore, Pakistan
azogomemon@hotmail.com

Abstract

In a set of n repeated Bernoulli trials where each trial results in an event E, the number of events materialized is a random variable. Even when the trials are not dependent, there do arise situations where it is tedious to find the probability distribution of this random variable. The paper expresses the moments of this random variable in terms of probabilities of particular events providing some of applications of this result.

Keywords: Bernoulli trials; rectangular lattice; horizontal and vertical joins.

1. Introduction

The use of Bernoulli trials often appears in ecological, epidemiological and agricultural research where events happen over space and time. The affliction of a disease, for instance, to human dwellings, agricultural plants and animals often creates a puzzling pattern that engages a scientist's concern, interest and curiosity. Linking of locations where events occur, may exhibit a peculiar fashion, but it is a phenomenon that scientists do not generally ignore for their understanding. They value its importance for various purposes. So when a group of dependent, or independent trials result in the same activity, or event, one may not limit his interest necessarily to the patterns of events randomly occurring. It could be in the number of specific patterns of these events as well. A researcher's eagerness may desire for information how the chances for this happening are distributed. The probability distribution of a random variable of this kind could be quite complex, depending on the kind of a configuration that a researcher considers.

Moran (1948), Fuchs and David (1965) investigate the distributions of certain configurations. Memon and David (1968) for instance, develop a theorem on factorial moments of the number of specific events that arise as a consequence of Bernoulli trials that happen simultaneously. They use this result to find the asymptotic distribution of the number of randomly occurring 'horizontal and vertical joins' at 'n^2' locations of an $n \times n$ lattice square; in this case, even for a join of this form and n is as large as 4, the derivation of the exact distribution of the number of such joins is not simple.

In view of complexities arising for the calculation of moments of the kind of variable indicated above, in this paper we develop a relationship between its moments and probabilities of particular events. Applications of this result are also given.

2. Main Result

Let $B_1, B_2, \ldots\ldots, B_n$ be Bernoulli trials so that each trial produces the event E with probabilities $p_1, p_2, \ldots., p_n$. We introduce the random variable X that denotes the number of events E that occur when these trials materialize simultaneously. In n trials, X can take a value 0, 1, 2, …, n with certain probabilities. We state the following theorem expressing the rth moment of X in terms of the probabilities of particular events in n trials.

Theorem 1

$$E(X^r) = a_{r1} \sum_{i_1} p_{i1} + a_{r2} \sum_{i_1 < i_2} p_{i1,\, i2} + \ldots\ldots + a_{rr} \sum_{i_1 < i_2 \ldots\ldots < i_r} p_{i1,\ldots\ldots,\, ir}$$

where $p_{i1,\ \ldots\ ,im}$ (= 0 for m > n) denotes the probability of m events in a particular subset of Bernoulli trials $B_{i1}, B_{i2}, \ldots\ldots, B_{im}$, and $i_1, i_2, \ldots, i_m$ = 1, 2, ….n. The coefficients a_{rm} of nonzero joint probabilities have the values

$$a_{r\,m} = \sum_{v=0}^{m-1} \binom{m}{v} (m - v)^r (-1)^v \quad ; \qquad m=1, 2, 3, \ \ldots, n$$

Proof

To prove the theorem let

φ_i = 1 when the trial B_i produces the event E
= 0 otherwise, i = 1, 2, ……, n.

so that the number of events that occur when all n trials are simultaneously run, can be expressed as

$$X = \varphi_1 + \varphi_2 + \ldots\ldots + \varphi_n \qquad (2.1)$$

which can take a value 0, 1, 2,……, n.

When r is a positive integer, from above it follows that

$$X^r = \sum_{i_1, i_2, \ldots\ldots i_{n-1}} \frac{r!}{(i_1)!\,(i_2)! \ldots\ldots (i_n)!} A \qquad (2.2)$$

where

$$A = \varphi_1^{i_1} \varphi_2^{i_2} \ldots\ldots\ldots \varphi_n^{i_n}$$

$i_n = r - i_1 - i_2 - \ldots. - i_{n-1}$; $i_1, i_2, \ldots, i_{n-1}$ = 0, 1, …, r

If some particular Bernoulli trials $B_{u1}, B_{u2}, \ldots, B_{um}$ result in E, we have $\varphi_{u1} = \varphi_{u2} = \ldots. = \varphi_{um} = 1$ with the associated probability $p_{u1,\ \ldots,\ um}$ for these m events. Note that this probability is zero for m > n. Expression (2.2) includes the sum of terms each with at least one variable φ_i and their products under the restriction $i_1 + i_2 + \ldots. + i_n = r$. Thus for $i_1 = r$, $i_2 = 0$, …, $i_n = 0$, the expression A includes the variable φ_1^r. The first term with similar powered random variables comprises

$$\varphi_1^r + \varphi_2^r + \ldots\ldots\ldots + \varphi_n^r$$

with expectation $\sum_i E\left(\varphi_i^r\right)$; i = 1, 2, ………,n

$$= a_{r1} \sum_i p_i \ ,$$

where $a_{r1} = 1$ (2.3)

that is, the first term involves probability $\mathbf{p_i}$ of E in the trial $\mathbf{B_i}$.

The product $\varphi_i^{i_1} \ \varphi_j^{r-i_1}$ involves two distinct variables in Expression (2.2) when $1 \le i_1 \le r-1$ for fixed i , j such that $1 \le i < j \le n$. The expectation $E\left(\varphi_i^{i_1} \ \varphi_j^{r-i_1}\right) = p_{ij}$ is constant for all $1 \le i_1 \le r-1$, so that its coefficient is

$$\sum_{i_1} \frac{r!}{(i_1)! \ (r-i_1)!}$$

simplifying to

$$a_{r2} = 2^r - 2. \qquad (2.4)$$

For fixed i,j,k the product $\varphi_i^{i_1} \ \varphi_j^{i_2} \ \varphi_k^{r-i_1-i_2}$ has different superscripts as i_1, i_2 vary such that

$$1 \le i_1 \le r-2, \ \ 1 \le i_2 \le r-i_1-1, \ \ 1 \le i < j < k \le n.$$

The expectation of this product is p_{ijk} (which is zero if n >3), the probability of three specific events in n Bernoulli trials. The coefficient of this probability in E(A) is

$$\sum_{i_1, i_2} \frac{r!}{i_1! \ i_2! \ (r-i_1-i_2)!}$$

and so summing over i_2 it is reduced to

$$= \sum_{i_1} \frac{r!}{i_1! \ (r-i_1)!} \left(2^{r-i_1} - 2\right)$$

that is,

$$a_{r3} = 3^r - 3 \ 2^r + 3 \qquad (2.5)$$

For n >3 we do not need coefficients a_{r4}, a_{r5}, ...

In (2.2) the expectation $E[\varphi_i^{i_1} \ \varphi_j^{i_2} \ \varphi_k^{i_3} \ \varphi_l^{r-i_1-i_2-i_3}]$ has the coefficient that is determined under the conditions

$$1 \le i_1 \le r-3, \quad 1 \le i_2 \le r-i_1-2, \quad 1 \le i_3 \le r-i_1-i_2-1, \ \ 1 \le i < j < k < l \le n.$$

The coefficient of probability of p_{ijkl}

$$a_{r4} = \sum_{i_1} \sum_{i_2} \sum_{i_3} \frac{r!}{i_1! \ i_2! \ i_3! \ (r-i_1-i_2-i_3)!}$$

$$= \sum_{i_1}\sum_{i_2} \frac{r!}{i_1!\, i_2!\, (r - i_1 - i_2)!} \left(2^{r - i_1 - i_2} - 2\right)$$

$$= \sum_{i_1} \frac{r!}{i_1!\, (r - i_1)!} \left(3^{r - i_1} - 3\, 2^{r - i_1} + 3\right)$$

$$= 4^r - 4\left(3^r\right) + 6\left(2^r\right) - 4 \tag{2.6}$$

Continuing the above procedure of obtaining the coefficients (2.3) to (2.6) from (2.2) after taking expectation we can determine the coefficient a_{rm} for m=1, 2, 3, …,r. It is based on m-1 summations.

$$a_{r\,m} = \sum_{v=0}^{m-1} \binom{m}{v} (m - v)^r (-1)^v \tag{2.7}$$

The last term is associated with r events E in n trials. It can be seen that the probability $p_{i1, \dots, ir}$ of r specific events has the coefficient (2.7) for m=r, which simplifies to r!.

2.1 REMARKS:

1) $E(X^r)$ in the statement of theorem applies to a positive integer r. However its value is 1 for r=0 in (2.2).

2) For r > n since the associated joint probability of r events E in n trials is zero, the coefficients a_{rm} for m > n are ignored in the theorem . Thus if n=2 and r is any positive integer > 2, the expression for the rth moment in the theorem contains only the first two terms. If each trial produces the event with the same p, then

$$E(X^r) = 2p + (2^r - 2)\, p^2.$$

3) From the above theorem we can derive the result on factorial moments that Memon and David obtain in their paper [3].

4) When the trials occur independently with equal probabilities p the above theorem assumes a simpler expression as

$$p_{i1, \dots, im} = p^m.$$

2.2 Corollary

For $r \le n$, we have

$$E(X) = \sum_i p_i \,,$$

$$E(X^2) = \sum_i p_i + 2 \sum_{i<j} p_{ij} \,,$$

$$E(X^3) = \sum_i p_i + 6 \sum_{i<j} p_{ij} + 6 \sum_{i<j<k} p_{ijk} \,,$$

$$E(X^4) = \sum_i p_i + 14 \sum_{i<j} p_{ij} + 36 \sum_{i<j<k} p_{ijk} + 24 \sum_{i<j<k<l} p_{ijkl}$$

the summations

$$\sum_{i} , \sum_{i<j} , \sum_{i<j<k} , \sum_{i<j<k<l}$$

extending over all particular sets of events of sizes 1, 2, 3, 4 in n trials respectively where i, j, k, l = 1, 2, …… n.

3. Applications of the Theorem

We now consider applications of this theorem in this section. The condition is that the integer $r \le n$.

Application I: Suppose that an event E can occur with probability p at a location during a period of time, and that there are n similar locations. Let X denote the number of events E that materialize at these locations simultaneously but independently. The moments of X in this case can be found by substituting

$$p_i = p, \quad p_{ij} = p^2, \quad p_{ijk} = p^3, \quad p_{ijkl} = p^4, \; \ldots\ldots\ldots.$$

in the above theorem. For instance, the fourth moment of X is

$$np\,[1 + 7(n - 1)p + 6(n - 1)(n - 2)p^2 + (n - 1)(n - 2)(n - 3)p^3]. \qquad (3.1)$$

Application II: Consider two independent sets of m and n locations. Let the probabilities that the events E and E* materialize at these locations be as follow.

Set A $\quad p_{a1}, p_{a2}, \ldots\ldots, p_{am}$ at Location 1, 2, ….., m

Set B $\quad p_{b1}, p_{b2}, \ldots\ldots, p_{bn}$ at Location 1, 2, ….. , n

Let X be the number of pairs of locations, one in Set A and the other in Set B, at which the events E and E* occur simultaneously. The number of possible single links is mXn each with probability $p_{ai}\, p_{bj}$. The moments of X can be determined by the above theorem using the probabilities of these links.

For example, if m = n = 2 and the probabilities of events E and E* are $p_{a1} = 0.2$, $p_{a2} = 0.4$, $p_{b1} = 0.1$, $p_{b2} = 0.3$ it is easy to see that X can take the values 0, 1, 2, 4. The first moment of X immediately comes to 0.24. It is found that the two, three and four links jointly materialize with probabilities 0.0548, 0.0096 and 0.0024. Therefore by the above corollary, the second, third and fourth moments of X are calculated as 0.3496, 0.6264, 1.4104 respectively.

On the contrary, it is cumbersome to first find the probability distribution of X even in the above a simple case. Here, for X=0 we consider seven possibilities that do not culminate in single links and the probability for this happening is 0.8076. For X=1, 2, 4 the number of possibilities that entail these links are four, four and one respectively. It can be shown that their probabilities are 0.1496, 0.0404 and 0.0024 respectively. With this information we can now find the moments.

Application III: Let n locations be arranged in some order, say horizontal. The two locations adjacent i^{th} and $(i+1)^{th}$ can be treated as a join when the event E occurs at each location independently with probability p during a specified time period. We define the random variables

$$\varphi_i = 1 \quad \text{for the } i^{th} \text{ join that materializes.} \quad i = 1, 2, \ldots, n-1. \qquad (3.2)$$
$$= 0 \quad \text{for otherwise,}$$

and X as the number of materializing specified joins when the n trials happen simultaneously. To find the first four moments of X for n > 10 the moments we have to find

$$\sum_{i}, \sum_{i<j}, \sum_{i<j<k}, \sum_{i<j<k<l}$$

that appear in the theorem. The first summation refers to the probability of single joins. Since the expectation of each φ_i in (3.2) is p^2 the first moment

$$E(X) = c_{11}\, p^2 \qquad (3.3)$$

where $c_{11} = n-1$

For the second summation we consider the combinations that produce two joins. This combination is possible when two joins are contiguous, or when the two single joins are separated. In the first case there are (n-2) possibilities each with probability p^3. For the second case the probability of this configuration is p^4 and the number of these configurations can be determined from the series

$$1 + 2 + \ldots\ldots\ldots\ldots + (n-4),$$

arising when forming all possible configurations of two disjoint links. Its sum is (n-3)(n-4)/2. Thus the total probability of two joins, adjacent or otherwise, when n trials happen simultaneously is

$$(n-2)\, p^3 + (1/2)\, (n-3)n-4)p^4 \,.$$

Now adding the first two terms in the expression for the second moment, as in the corollary, we have

$$E\,(X^2) = c_{21}\, p^2 + c_{22}\, p^3 + c_{23}\, p^4 \qquad (3.4)$$

where
$c_{21} = (n - 1).$
$c_{22} = 2(n - 2).$
$c_{23} = (n - 3)\,(n - 4).$

Similarly, the probability of three links involves p^4, p^5 and p^6 and is simplified to

$$(n-3)p^4 + (n-4)(n-5)\, p^5 + (1/6)\, (n-5)n-6)p^6 \,.$$

The evaluation of fourth moment depends on four links forming configurations that involve p^5, $p^{6,}$ p^7, p^8.

Hence for n>10, the third and fourth moments of X are:

$$E(X^3) = c_{31}p^2 + c_{32}\, p^3 + c_{33}\, p^4 + c_{34}\, p^5 + c_{35}\, p^6. \quad (3.5)$$

where
$c_{31} = n - 1$, $c_{32} = 6(n - 2).$
$c_{33} = 3(n - 2)\,(n - 3)$, $c_{34} = 6(n - 4)\,(n - 5).$
$c_{35} = (n - 5)(n - 6)\,(n - 7).$

and

$$E(X^4) = c_{41}\, p^2 + c_{42}\, p^3 + c_{43}\, p^4 + c_{44}\, p^5 + c_{45}\, p^6 + c_{46}\, p^7 + c_{47}\, p^8 \qquad (3.6)$$

where $c_{41} = n - 1$, $c_{42} = 14(n - 2)$.
$c_{43} = (n - 3)(7n + 8)$, $c_{44} = 12(n - 4)(3n - 13)$.
$c_{45} = 6(n - 1)(n - 5)(n - 6)$, $c_{46} = 12(n - 8)(n - 6)(n - 76)$.
$c_{47} = (n - 7)(n - 8)(n - 9)(n - 10)$.

We can use the above moments (3.3),…..,(3.6) to find the asymptotic distribution of the random variable X. Assuming that $np^2 \rightarrow \lambda$ as $n \rightarrow \infty$, and p is small, it is easy to see that the first, second, third and fourth moments of X are simplified to

$$\begin{aligned} &\lambda, \\ &\lambda^2 + \lambda, \\ &\lambda^3 + 3\lambda^2 + \lambda, \\ &\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda, \end{aligned} \qquad (3.7)$$

respectively, indicating that X is asymptotically distributed as a Poisson random variable with λ as its parameter.

It may be noted that the theorem in the above application for n = 4, for instance, provides first four moments as:

$$E(X) = c_{11}\, p^2 \qquad (3.8)$$

$$E(X^2) = c_{21}\, p^2 + c_{22}\, p^3$$

$$E(X^3) = c_{31} p^2 + c_{32}\, p^3 + c_{33}\, p^4 .$$

$$E(X^4) = c_{41}\, p^2 + c_{42}\, p^3 + c_{43}\, p^4$$

The higher moments in this case involve p^2, p^3 and p^4 with different coefficients.

***Application IV*:** Moran (1948), Memon and David (1968) determine factorial moments of the randomly emerging number of horizontal and vertical *BB* joins of a rectangular lattice based on *m* x *n locations* when the event B can independently occur at each *location* with probability p during a given time period. They found the moments of this random variable using their theorem on factorial moments (as mentioned in the above remark). The proposed theorem in this paper can be applied to directly determine the moments of their random variable.

4. Reference

1. Fuchs, C.E. and David, H.T. (1965). Poisson Limits of Multivariate Run Distributions: Ann. Math. Stat., 36: 215-225.

2. Hogg, R.K and Craig, A.T (1978). Introduction to Mathematical Statistics, Macmillan Publishing Company, New York.

3. Memon, A.Z. and David, H.T. (1968). The Distribution of Lattice Join Counts, Bulletin of the Institute of Statistical Research and Training, Dhaka, Vol. 2, No. 2 - Reprint Series No. 252, Statistical Laboratory, Iowa State University, USA

4. Moran, P.A.P. (1948): Interpretation of Statistical Maps, J.R. Statist. Soc., 10: 243 – 251.