# A Characterization of Erlang-truncated Exponential distribution in Record Values and its use in Mean Residual Life

Muhammad Mohsin
Department of Mathematics
COMSATS Institute of Information Technology
Lahore, Pakistan

Saman Shahbaz
Department of Mathematics
COMSATS Institute of Information Technology
Lahore, Pakistan
samans@ciitlahore.edu.pk

Muhammad Qaiser Shahbaz
Department of Mathematics
COMSATS Institute of Information Technology
Lahore, Pakistan
qshahbaz@gmail.com

## Abstract

In this paper the Erlang-truncated exponential distribution is characterized by the conditional expectation of the record values. The utility of our result is demonstrated in mean residual life by using this characterization.

**Keywords:** Erlang-truncated exponential distribution, Record values, Mean residual life

## 1. Introduction

Consider a sequence of real numbers $\{X_1, X_2, .....\}$, which are independently and identically, distributed with common cumulative distribution $F(x)$. Let $Y_i = \max(\min)\{X_1, X_2, ......X_i\}$ for $i \geq 1$, then $X_j$ is called an upper (lower) record value of $\{X_i, i \geq 1\}$ if $Y_j > Y_{j-1}, j > 1, (Y_j < Y_{j-1}, j > 1)$. It is easy to see that $X_1$ is an upper as well as a lower record.

The properties and the characterizations of the record values have been extensively studied in literature. (See Arnold et al. [4], Novzorov [8], Raqab [9] and Ahsanullah [2, 3]).

A random variable X is said to have Erlang-truncated exponential distribution (see Alosey [5]), if its probability density function is of the form

$$f(x) = \beta\alpha_\lambda e^{-\beta x(\alpha_\lambda)}, \quad 0 \leq x < \infty, \beta > 0, \lambda > 0, \qquad (1.1)$$

Where $\alpha_\lambda = \left(1 - e^{-\lambda}\right)$

The distribution function is

$$F(x) = 1 - e^{-\beta x(\alpha_\lambda)}. \tag{1.2}$$

The pdf of the nth upper record (see Ahsanullah [1], Nagaria [7]) is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \qquad -\infty < x < \infty, \tag{1.3}$$

where $R(x) = -ln\left[1 - F(x)\right], 0 \le F(x) \le 1.$

The joint pdf of n upper records, $X_{U(1)}, X_{U(2)}, ..., X_{U(n)}$ is

$$f(x_1, x_2, ...x_n) = \prod_{i=1}^{n-1} r(x_i) f(x_n), \qquad -\infty < x_1 < .... < x_n < +\infty, \tag{1.4}$$

where $r(x) = \frac{d}{dx} R(x).$

The joint distribution of $X_{U(k)}$ and $X_{U(m)}$ is

$$f(x_k, x_m) = \frac{\left[R(x_k)\right]^{k-1}}{(k-1)!} r(x_K) \frac{\left[R(x_m) - R(x_k)\right]^{(m-k-1)}}{(m-k-1)!} f(x_m), \tag{1.5}$$

$$-\infty < x_k < x_m < \infty.$$

The conditional pdf of $X_{U(m)} \big| X_{U(k)} = x_k$ is

$$f\left(x_m \mid X_{U(k)} = x_k\right) = \frac{\left[R(x_m) - R(x_k)\right]^{(m-k-1)}}{(m-k-1)!} \frac{f(x_m)}{1 - F(x_k)}, \tag{1.6}$$

$$-\infty < x_k < x_m < \infty.$$

The distributions in (1.3), (1.5) and (1.6) can be obtained for any probability distribution. Mohsin [6] has obtained the recurrence relation for single and product moments of record statistics for distribution (1.1). In this paper we have obtained a characterization of distribution (1.1) in terms of conditional moments of the record values. Section 2 shows a necessary and sufficient condition to identifying the distribution for different situations. Section 3 illustrates the usefulness of this result by analyzing the mean residual life property.

## 2. Characterization based on the moments of the record values

In this section a characterization is presented by the conditional expectation of the record values:

**Theorem:**

Suppose $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables having continuous cumulative distribution function $F(x)$, then

$$E\left[X_{U(n)} \mid X_{U(m)} = x\right] = \frac{1}{\beta\alpha_\lambda}\left[(n-m) + \beta x\alpha_\lambda\right], \ \beta > 0, \lambda > 0 \qquad (2.1)$$

holds for $n = m+1$ and $n \geq m+2$ iff X has a p.d.f (1.1).

**Proof:**

The conditional p.d.f. $X_{U(n)} = y$ given $X_{U(m)} = x (n > m)$ is given by

$$f_{n,m}(y \mid x) = \frac{1}{(n-m-1)!}\left[-ln\{1-F(y)\} + ln\{1-F(x)\}\right]^{n-m-1} \frac{f(y)}{1-F(x)}, \text{for } x < y.$$

Now,

$$E\left[X_{U(n)} \mid X_{U(m)} = x\right] = \int_x^\infty \frac{y}{(n-m-1)!}\left[-ln\{1-F(y)\} + ln\{1-F(x)\}\right]^{n-m-1} \frac{f(y)}{1-F(x)} dy.$$

$$= \frac{\beta\alpha_\lambda}{(n-m-1)!}\int_x^\infty y\left[\beta y\alpha_\lambda - \beta x\alpha_\lambda\right]^{n-m-1} e^{-[\beta y\alpha_\lambda - \beta x\alpha_\lambda]} dy.$$

Using the transformation $(\beta y\alpha_\lambda - \beta x\alpha_\lambda) = t$ and after simplification, we get

$$E\left[X_{U(n)} \mid X_{U(m)} = x\right] = \frac{1}{\beta\alpha_\lambda}\left[(n-m) + \beta x\alpha_\lambda\right].$$

Now we prove the necessity part. If (2.1) holds, $E(X) < \infty$, then we have

$$\int_x^\infty \frac{y}{(n-m-1)!}\left[-ln\{1-F(y)\} + ln\{1-F(x)\}\right]^{n-m-1} f(y) dy = \frac{1}{\beta\alpha_\lambda}\left[(n-m) + \beta x\alpha_\lambda\right]\left[1-F(x)\right]. \qquad (2.2)$$

**Case 1**

When $n = m+1$, equation (2.2) reduces as

$$\int_x^\infty y f(y) dy = \left[\frac{1+\beta x\alpha_\lambda}{\beta\alpha_\lambda}\right]\left[1-F(x)\right]. \qquad (2.3)$$

Differentiating (2.3) w.r.to $x$, we get

$$-xf(x)=\left[\frac{1+\beta x\alpha_\lambda}{\beta\alpha_\lambda}\right](-f(x))+[1-F(x)]. \tag{2.4}$$

### Case 2

When $n\geq m+2$, differentiate (2.2) w.r.to $x$, we get

$$\begin{aligned}&-\int_x^\infty\frac{y}{(n-m-2)!}\left[-ln\{1-F(y)\}+ln\{1-F(x)\}\right]^{n-m-2}\frac{f(x)f(y)}{1-F(x)}dy=\\&-\left[\frac{(n-m)+\beta x\alpha_\lambda}{\beta\alpha_\lambda}\right]f(x)+[1-F(x)].\end{aligned} \tag{2.5}$$

If $n=m+2$, we let $n=m+1$ in equation (2.2), then (2.5) can be written as

$$-\int_x^\infty\frac{yf(x)f(y)}{1-F(x)}dy=-\left[\frac{2+\beta x\alpha_\lambda}{\beta\alpha_\lambda}\right]f(x)+[1-F(x)]. \tag{2.6}$$

Equation (2.4) and (2.6) provide the following differential equation.

$$\frac{-f(x)}{\beta\alpha_\lambda}+[1-F(x)]=0 \tag{2.7}$$

Substitution $y=1-F(x), y'=-f(x)$, the equation (2.7) reduces to

$$\frac{y'}{\beta\alpha_\lambda}+y=0 \tag{2.8}$$

Differential equation in (2.8) yield the following solution

$y=Ae^{-\beta x\alpha_\lambda}$ , where A is a constant.

Considering $F(0)=0$, A is determined to be equal to 1. So

$$1-F(x)=e^{-\beta x\alpha_\lambda}.$$

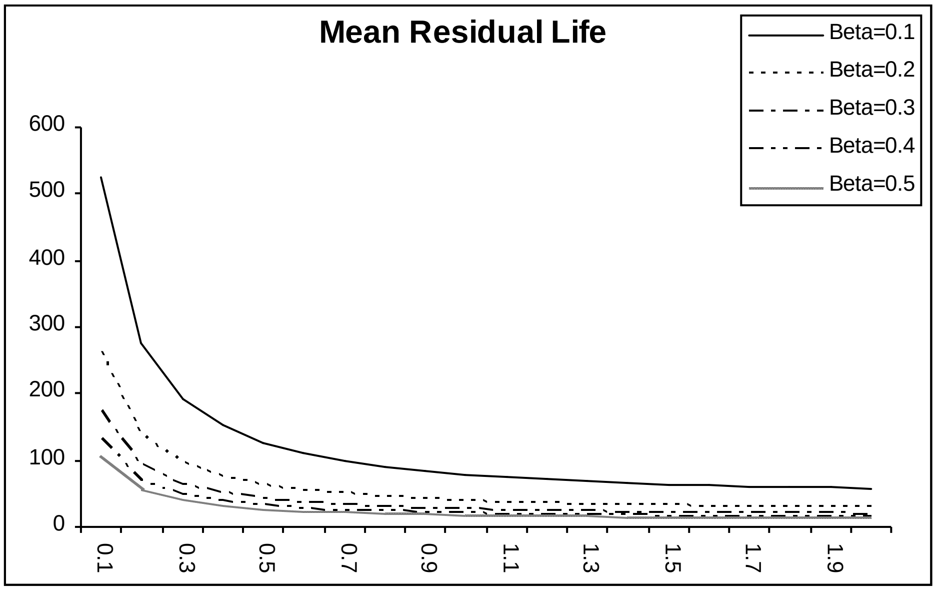
which is the desired result.

## 3. Application of the Conditional Expectation

In this section we have presented the application of the above results in the theory of reliability of *n*th record given the information of *m*th record. The average residual life of *n*th record given the information of *m*th record is given as $E\left[X_{U(n)}\mid X_{U(m)}=x\right]-x$. Now using (2.1) we have:

$$E\left[X_{U(n)}\mid X_{U(m)}=x\right]-x=(n-m)/\beta\alpha_\lambda \tag{3.1}$$

From (3.1) we can see that the mean residual life of *n*th record does not depends upon the *m*th record. The mean residual life just depends upon the parameters and the difference $(n-m)$. The graph of (3.1) for various values of $\beta$ and $\lambda$ is given below:




From above graph we can readily see that the mean residual life of *n*th record given the information of *m*th record decreases exponentially with increase in the value of $\lambda$. This indicates that the Earlang–truncated Exponential distribution with larger value of $\lambda$ will have smaller mean residual life of *n*th record.

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