Design of Attribute Control Chart Based on Regression Estimator

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Abstract

This paper presents a statistical analysis control chart for nonconforming units in quality control. In many situations the Shewhart control charts for nonconforming units may not be suitable or cannot be used. For many processes, the assumptions of binomial distribution may deviate or may provide inadequate model. In this Study we propose a new control chart, P_r chart, which is based on regression estimator of proportion based on single auxiliary variable. The performance of the compared with *P* and *Q* charts with probability to signal as a performance measure. It has been observed that the proposed chart is superior to the P and Q charts. This study will help quality practitioners to choose an efficient alternative to the classical P and Q charts for monitoring nonconforming units in industrial process.

Keyword: Nonconforming units, Attribute control charts, Probability limits.

1. Introduction

Control Charts are commonly used in monitoring and detecting shifts in the production processes. Many quality characteristic cannot be conveniently measured numerically. In such cases, we usually classify each item inspected as either conforming or nonconforming to the specification on the quality characteristics. When an item is produced or purchased, it is inspected in order to identify if it satisfies a number of specifications. An item that does not satisfy those specifications is called a defective or a non-conforming item. These defects lead to rework or they are characterized as scrap or second quality product. In any case we have a loss of money or working time or both. In order to avoid such products, control charts for the characteristics (attributes) have been developed. The *P*-chart, developed by Shewart (1924) is widely used to monitor the fraction of non-confirming units.

The literature on control charts provides a variety of control charts to monitor the fraction nonconforming of the process. To refer a few of these: Quesenberry (1991) presented that the Arcsine *P*-chart gives a better approximation to the nominal lower tail area. Acosta (1999) considered improved *P* charts to monitor process quality. Shore (2000) proposed the general control charts for attributes.

More details about the Shewart control chat can be seen in Niaki and Abbasi (2007), Mehmood et al. (2013) and Riaz et al. (2014).

According to Montgomery (2003), "in many quality control environments, the process or product under consideration has two or more correlated quality characteristics. For example, the quality of a chemical process may be a function of process temperature, pressure, and flow rates, all of which need to be monitored in a situation where some correlation may exist between any two of them. In these cases, if we want to monitor these quality characteristics separately, there will be some error associated with the out of control detection procedure". Riaz (2008) proposed the control charts using the regression estimator for variable sampling.

According to the best of the authors knowledge there is no work in the area of quality control using the proportion estimator for the attribute quality characteristics.

In this paper, we will propose attribute control chart using the proportion estimator given by Das (1982). We will develop the control limits by following Riaz (2008). We will propose a proportion control chart namely the Pr chart, based on Das (1982) based estimate of proportion.

2. Proposed Control Charts and Construction of Control Limits

2.1 Regression Estimator for Proportion:

Assuming bivariate normality of (x, y) a Shewhart-type process proportion of nonconforming control chart, namely chart (proposed which is based on the regression type estimator) of process proportion level. The regression estimator for proportion of Y using a proportion of single auxiliary variable X. I assume the numbers '$1, 2, ..., n$' of (y_1, x_1), $(y_2, x_2), \ldots, (y_n, x_n)$ is the proportion of a bivariate random sample given as:

$$P_r = p_i + b\left(P_j - p_j\right), \tag{1}$$

Where $b = r_{ij}\left(S_i / S_j\right)$ (2)

Where P_r the population proportion of Y, p_i is the sample proportion of Y, P_j is the population proportion of X and p_j is sample proportion of X. Also, r_{ij} is the sample correlation between X and Y and S_i and S_j are corresponding sample standard deviations.

2.2 Proposed Control Charts and Construction of Control Limits

Assume that (P_i, P_j) are bivariate normally distributed. Suppose the relationship between P_i and Pr be defined by a random variable C as follows [Riaz (2008)].

$$C = \sqrt{n}(P_r - P_i)/\sigma_i \tag{3}$$

Where $\sigma_i = \sqrt{\frac{P_i Q_i}{n}}$ and $Q_i = 1 - P_i$

The relationship defined in (3) helps in determining the parameters (i.e. centerline, lower and upper control limits) of the proposed *Pr* chart. Now, if the distributional behavior of *C* is known then the sample statistic *Pr* can easily be used for the testing of hypotheses about shifts in P_i. When (P_i, P_j) follow bivariate normal distribution, the distributional behavior of *C* depends only on ρ_{ij} (the correlation between P_i and P_j) and *n*. The distributional behavior of *C*, in terms of its proportion and quantile points, is required for the development of the *Pr* chart, and is explored in the following paragraphs when (P_i, P_j) follow a bivariate normal distribution.

By taking expectations of Eq. (3), we have

$$E(C) = \sqrt{n}E(P_r - P_i)/\sigma_i \tag{4}$$

Note that $E(P_r)$ can be replaced by $\bar{P}_r$ as discussed in Hillier (1969).

Then simplifying and rearranging equation (4), we get the following results:

$$\hat{P}_i = \bar{P}_r - \hat{\sigma}_i\, E(C)/\sqrt{n} \tag{5}$$

The regression estimator *Pr* is generally a biased estimator of the population proportion but the bias vanishes when the relationship between P_i and P_j is linear (see Sukhatme and Sukhatme, 1984, p. 238). So, for the case of bivariate normal distribution (P_i, P_j), *Pr* is unbiased for P_i and hence *E*(*C*) =0. Thus, (5) results into the following:

$$\hat{P}_i = \bar{P}_r \tag{6}$$

Replacing the estimate of P_i in (6) we have the following results:

$$E(P_r) \cong \bar{P}_r \tag{7}$$

For standard error, let the standard deviation of *C be*

$$\sigma_C = k \tag{8}$$

It is not easy to get the analytical results for k because $E(P_r^2)$ is difficult to obtain analytically. So, simulation methods are often used to evaluate the expectation of a statistic, see Ross (1990).

By taking the variance on of C and by simplifying, we have the following results:

$$\sigma_C = \sqrt{n}\,\sigma_{Pr}/\sigma_i \tag{9}$$

where σ_{Pr} represents the standard deviation of distribution of sample statistic Pr.

Using (8) in (9), rearranging and substituting the estimate for σ_i, we get the following:

$$\hat{\sigma}_{Pr} = k\,\hat{\sigma}_i/\sqrt{n} \tag{10}$$

An approximation for σ_{Pr}, when (P_i, P_j) follows a bivariate normal distribution, is given as (see Sukhatme and Sukhatme, 1984, p. 267):

$$\hat{\sigma}_{Pr} = \frac{1}{n}\sqrt{P_i Q_i\left(1 - \rho_{ij}^2\right)(1 + 1/n - 3)} \tag{11}$$

Consequently,

$$k \cong \sqrt{\left(1-\rho_{ij}^{2}\right)(1+1/n-3)} \tag{12}$$

Where ρ_{ij} is the correlation coefficient between y and x.

2.3 Simulation Study

The quantile points of the distribution of *C*, let *Ca* represents the *a*th quantile point of the distribution of *C* (i.e. the point where *C* completes *a*% area). The analytical results for *Ca* are difficult to obtain; so, the simulation results are obtained for *Ca*. For a bivariate normal distribution of (P_i, P_j) the quantile points of the distribution of *C* depends entirely on ρ_{ij} and *n*. Using the bivariate normal distribution 10,000 simulated random samples, the results of *Ca* have been obtained.

Based on these results, the values of some commonly used quantile points, are provided for *n*=10, 20. . . 1000 in Appendix Tables A2–A11 at some representative values of ρ_{ij}. The similar results can easily be obtained for any combination of ρ_{ij} and *n*. These quantile points help determining the control limits and the power of the proposed *Pr* chart to detect shifts in process of proportion of the defectives.

3. Parameters of the Proposed Chart

Finally, the control limits for the proposed control chart is given as

$$LCL = \bar{P}_r - 3\sigma_{Pr} \qquad CL = \bar{P}_r \qquad UCL = \bar{P}_r + 3\sigma_{Pr}$$

By using results

$$LCL = \bar{P}_r - 3c\,{\hat{\sigma}_i}/{\sqrt{n}} \qquad CL = \bar{P}_r \qquad UCL = \bar{P}_r + 3c\,{\hat{\sigma}_i}/{\sqrt{n}}$$

The use of 3-sigma limits is based on the symmetric assumption of the plotted statistic, we will see that the distribution of Pr is not symmetric at least for small to moderate values of n. Hence there is a need to develop the probability limit structure for the proposed chart. Probability limits for the proposed chart can be computed by using quantile points of the distribution of C.

Let α be the specified probability of making Type-I error, denoting α-quantile of the distribution of C by C_α. The probability limits based on Pr are given as:

$$LCL = P_{rl} \qquad \text{with} \qquad P_n(P_r = P_{rl}) \le \propto_l$$

$$UCL = P_{ru} \qquad \text{with} \qquad P_n(P_r = P_{ul}) \ge 1 - \propto_u$$

Where $\propto = \propto_l + \propto_u$ and *Pn* represents the cumulative distribution function for a given value of *n*.

Now after simplification, we have the following results.

$$LCL = P_{rl} = \bar{P}_r + C_l\,\hat{\sigma}_i/\sqrt{n} \qquad \text{with} \qquad P_n(c = c_l) \le \propto_l$$

$$UCL = P_{ul} = \bar{P}_r + C_u\,\hat{\sigma}_i/\sqrt{n} \qquad \text{with} \qquad P_n(c = c_u) \ge 1 - \propto_u.$$

We need to find the results of the following by using simulation method.

$$LCL = \bar{P}_r + C_{0.01}\,\hat{\sigma}_i/\sqrt{n} \qquad CL = \bar{P}_r \qquad UCL = \bar{P}_r + C_{0.99}\,\hat{\sigma}_i/\sqrt{n}$$

The values of *k* are provided in Appendix Table1 for *n*=10, 20, 100, 500, 1000. Asymptotically, *C* is normally distributed, from Appendix Tables A2–A11.

4. Comparison

In this paper the performance of the *Pr* chart is compared with *P* (conventional Shewhart attribute control chart) and *Q* chart for binomial data developed by Quesenberry (1991). The efficiency of *Pr* chart as compared to *P* and *Q* chart has been examined using power curves as a performance measure. Using their respective control structures, the power curves for different combinations of ρ_{ij} have been constructed.

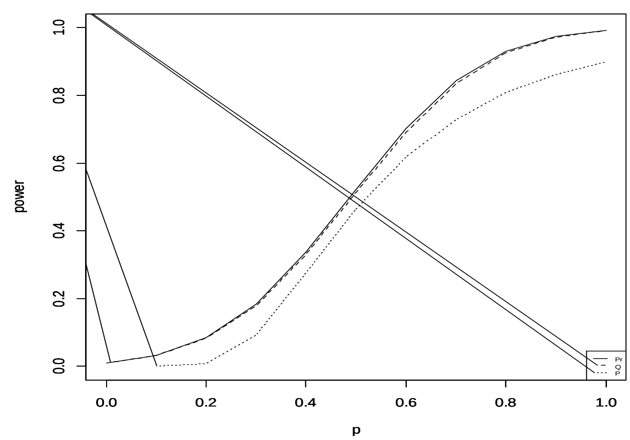

Figure 1: Power curves of Pr, P and Q charts for $\rho_{ij} = 0.30$

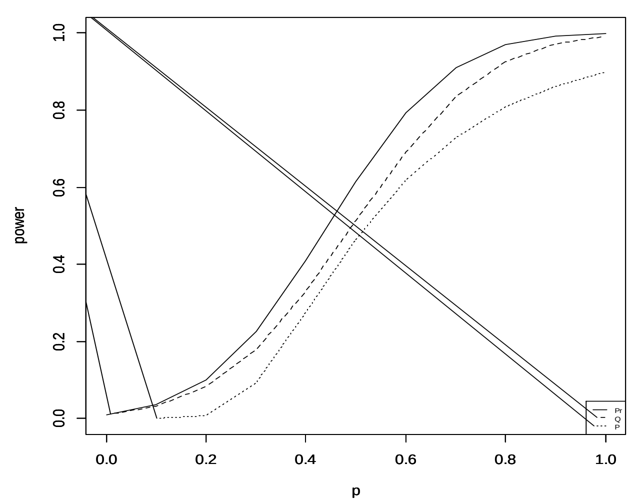

Figure 2: Power curves of Pr, P and Q charts for $\rho_{ij} = 0.60$

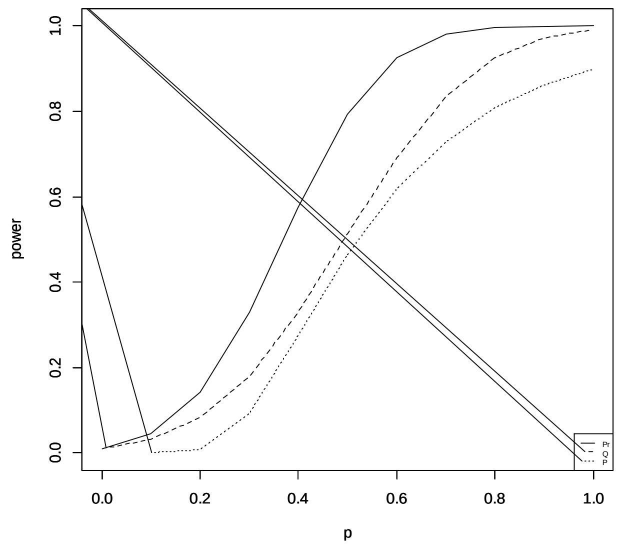

Figure 3: Power curves of Pr, P and Q charts for $\rho_{ij} = 0.90$

Figures 1-3 shown that the power curve of Pr chart is almost equally powerful as Q chart and more powerful for $\rho_{ij} = 0.30$, and more powerful then Q and P chart for $\rho_{ij} = 0.60\ and\ \rho_{ij} = 0.90$.

This show that the *Pr* chart has higher probability to signal shifts in process of non-confirming items as compared to both *P* and *Q* chart.

5. Conclusion

The attribute control charts are particularly useful in the service industries and in non-confirming quality improvement efforts. The classical application of *P* chart requires that the parameters of the distribution are known. In many situations the true fraction nonconforming, *P*, is unknown and need to be estimated. This study proposes an efficient control chart, namely *Pr* chart, to monitor the process proportion or the non-confirming items, based on proportion regression estimator. We derive the parameters of the proposed chart. The performance of the proposed chart is compared with the classical *P* and *Q* charts using OC curves. It has been shown that Pr chart more efficient to the *P* and *Q* chart. The design of the *Pr* chart is established and is shown to be more efficient as compared to the classical *P* chart and *Q* chart, particularly for bivariate data.

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6. References

1. Acosta, M., Pignatiello JJ, Rao, B.V. (1999). A comparison of control charting procedures for monitoring process dispersion. *IIE Trans*ition. 31,569–579.

2. Das, A. K. (1982). On the use of auxiliary information in estimating proportions. *Journal of Indian Statistical Association*, 20, 99-108.

3. Hillier, F. S. (1969). $\bar{X}$ and R chart control limits based on a small number of subgroups, *Journal of Quality Technology,* 1, 17–26.

4. Mehmood, R., Riaz, M., & Does, R. J. M. M. (2013). Control charts for location based on different sampling schemes. *Journal of Applied Statistics*, 40(3), 483-494.

5. Montgomery, D. C. (2003). "Introduction to Statistical Quality Control." 5th ed., John Wiley & Sons, New York, NY.

6. Niaki, S. T. A., & Abbasi, B. (2007). On the monitoring of multi-attributes high-quality production processes. *Metrika*, *66*(3), 373-388.

7. Quesenberry, C.P. (1991). SPC Q charts for binomial parameter p: Short or long runs. *Journal of Quality Technology,* 23, 239-246.

8. Riaz, M. (2008). Monitoring process mean level using auxiliary information. Statistica Neerlandica, 62(4), 458–481.

9. Riaz, Muhammad, Ahmad, S. Riaz, M., Abbasi, S. A. and Lin, Z. Y. (2014). On Efficient Median Control Charting. Journal of the Chinese Institute of Engineers. *Journal of the Chinese Institute of Engineers*, 37, 358-375

10. Ross, S. M. (1990). A course in simulations, *Macmillan Publishing Co. New York.*

11. Shewhart, W.A. (1924). Some applications of statistical methods to the analysis of physical and engineering data. *Bell System Technical Journal,* 3(1), 43-87.

12. Shore, H (2000). General control charts for attributes. *IIE Transactions* 32, 1 149-1 160.

13. Sukhatme, P. V. and B. V. Sukhatme (1984). Sampling theory of surveys with applications, *Third Edition. Iowa State University Press, IA.*

7. Appendix

Table 1: Control chart coefficient k of the Pr chart ρ_{ij}

n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
10	1.064	1.047	1.019	0.979	0.925	0.855	0.763	0.641	0.466	0.151
20	1.024	1.008	0.981	0.943	0.891	0.823	0.735	0.617	0.449	0.145
30	1.013	0.997	0.971	0.933	0.881	0.815	0.727	0.611	0.448	0.143
40	1.008	0.992	0.966	0.928	0.877	0.810	0.724	0.608	0.443	0.142
50	1.006	0.990	0.964	0.926	0.875	0.808	0.721	0.606	0.441	0.142
60	1.004	0.988	0.962	0.924	0.873	0.806	0.720	0.605	0.439	0.142
70	1.002	0.987	0.961	0.923	0.872	0.805	0.719	0.604	0.439	0.142
80	1.001	0.986	0.960	0.922	0.871	0.805	0.719	0.603	0.439	0.141
90	1.000	0.985	0.959	0.922	0.870	0.804	0.718	0.603	0.438	0.141
100	1.000	0.984	0.958	0.921	0.870	0.804	0.717	0.603	0.438	0.141
500	0.996	0.981	0.953	0.917	0.866	0.800	0.714	0.600	0.436	0.141
1000	0.995	0.980	0.954	0.916	0.866	0.800	0.714	0.600	0.436	0.141

Table 2: Quantile points of the distribution of Q (when $\rho_{ij} = 0.1$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-2.47450	-1.74960	-1.36316	-0.89522	-0.71744	0.717445	0.895221	1.363169	1.749608	2.474504
20	-2.38179	-1.68405	-1.31209	-0.86168	-0.69056	0.690565	0.861680	1.312096	1.684056	2.381793
30	-2.35716	-1.66664	-1.29852	-0.85276	-0.68342	0.683423	0.852769	1.298526	1.666641	2.357162
40	-2.34575	-1.65857	-1.29224	-0.84864	-0.68011	0.680117	0.848643	1.292244	1.658578	2.345758
50	-2.33918	-1.65392	-1.28862	-0.84626	-0.67821	0.678210	0.846264	1.288621	1.653928	2.339182
60	-2.33490	-1.65090	-1.28626	-0.84471	-0.67697	0.676970	0.844716	1.286264	1.650902	2.334903
70	-2.33189	-1.64877	-1.28460	-0.84362	-0.67609	0.676098	0.843628	1.284608	1.648777	2.331897
80	-2.32966	-1.64720	-1.28338	-0.84282	-0.67545	0.675452	0.842822	1.283381	1.647202	2.329669
90	-2.32795	-1.64598	-1.28243	-0.84220	-0.67495	0.674954	0.842201	1.282435	1.645988	2.327952
100	-2.32658	-1.64502	-1.28168	-0.84170	-0.67455	0.674559	0.841708	1.281684	1.645023	2.326588
200	-2.32055	-1.64075	-1.27836	-0.83952	-0.67281	0.672810	0.839525	1.278360	1.640757	2.320554
300	-2.31858	-1.63936	-1.27727	-0.83881	-0.67223	0.672237	0.838811	1.277273	1.639362	2.318580
400	-2.31760	-1.63866	-1.27673	-0.83845	-0.67195	0.671953	0.838456	1.276733	1.638669	2.317600
500	-2.31701	-1.63825	-1.27641	-0.83824	-0.67178	0.671783	0.838244	1.276410	1.638254	2.317014
800	-2.31613	-1.63763	-1.27592	-0.83792	-0.67152	0.671529	0.837927	1.275927	1.637635	2.316139
1000	-2.31584	-1.63742	-1.27576	-0.83782	-0.67144	0.671445	0.837822	1.275767	1.637429	2.315847

Table 3: Quantile points of the distribution of Q (when $\rho_{ij} = 0.2$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-2.43672	-1.72289	-1.34235	-0.88155	-0.70649	0.706491	0.881552	1.342356	1.722895	2.436723
20	-2.34542	-1.65834	-1.29206	-0.84852	-0.68002	0.680021	0.848523	1.292062	1.658344	2.345428
30	-2.32117	-1.64119	-1.27870	-0.83974	-0.67298	0.672989	0.839748	1.278701	1.641194	2.321172
40	-2.30994	-1.63325	-1.27251	-0.83568	-0.66973	0.669733	0.835686	1.272514	1.633254	2.309943
50	-2.30346	-1.62867	-1.26894	-0.83334	-0.66785	0.667855	0.833343	1.268947	1.628675	2.303467
60	-2.29925	-1.62569	-1.26662	-0.83181	-0.66663	0.666634	0.831819	1.266626	1.625696	2.299253
70	-2.29629	-1.62360	-1.26499	-0.83074	-0.66577	0.665775	0.830748	1.264995	1.623603	2.296293
80	-2.29409	-1.62205	-1.26378	-0.82995	-0.66513	0.665139	0.829954	1.263786	1.622052	2.294099
90	-2.29240	-1.62085	-1.26285	-0.82934	-0.66464	0.664649	0.829342	1.262855	1.620857	2.292408
100	-2.29106	-1.61990	-1.26211	-0.82885	-0.66426	0.664260	0.828856	1.262115	1.619907	2.291065
200	-2.28512	-1.61570	-1.25884	-0.82670	-0.66253	0.662537	0.826707	1.258842	1.615706	2.285124
300	-2.28318	-1.61433	-1.25777	-0.82600	-0.66197	0.661973	0.826004	1.257771	1.614332	2.283180
400	-2.28221	-1.61364	-1.25723	-0.82565	-0.66169	0.661694	0.825654	1.257239	1.613649	2.282215
500	-2.28163	-1.61324	-1.25692	-0.82544	-0.66152	0.661526	0.825446	1.256922	1.613241	2.281638
800	-2.28077	-1.61263	-1.25644	-0.82513	-0.66127	0.661276	0.825134	1.256446	1.612632	2.280776
1000	-2.28048	-1.61242	-1.25628	-0.82503	-0.66119	0.661193	0.825030	1.256289	1.612429	2.280489

Table 4: Quantile points of the distribution of Q (when $\rho_{ij} = 0.3$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-2.37241	-1.67742	-1.30693	-0.85828	-0.68784	0.687847	0.858288	1.306931	1.677428	2.372419
20	-2.28353	-1.61458	-1.25796	-0.82613	-0.66207	0.662076	0.826131	1.257965	1.614581	2.283532
30	-2.25991	-1.59788	-1.24495	-0.81758	-0.65522	0.655229	0.817588	1.244956	1.597883	2.259917
40	-2.24898	-1.59015	-1.23893	-0.81363	-0.65205	0.652059	0.813632	1.238933	1.590153	2.248984
50	-2.24267	-1.58569	-1.23545	-0.81135	-0.65023	0.650231	0.811351	1.235459	1.585695	2.242679
60	-2.23857	-1.58279	-1.23320	-0.80986	-0.64904	0.649041	0.809867	1.233200	1.582794	2.238576
70	-2.23569	-1.58075	-1.23161	-0.80882	-0.64820	0.648206	0.808824	1.231612	1.580757	2.235694
80	-2.23355	-1.57924	-1.23043	-0.80805	-0.64758	0.647586	0.808052	1.230435	1.579246	2.233558
90	-2.23191	-1.57808	-1.22952	-0.80745	-0.64710	0.647109	0.807456	1.229528	1.578082	2.231912
100	-2.23060	-1.57715	-1.22880	-0.80698	-0.64673	0.646730	0.806983	1.228808	1.577158	2.230604
200	-2.22482	-1.57306	-1.22562	-0.80489	-0.64505	0.645053	0.804890	1.225621	1.573068	2.224820
300	-2.22292	-1.57173	-1.22457	-0.80420	-0.64450	0.644504	0.804206	1.224579	1.571730	2.222927
400	-2.22198	-1.57106	-1.22406	-0.80386	-0.64423	0.644232	0.803866	1.224061	1.571065	2.221988
500	-2.22142	-1.57066	-1.22375	-0.80366	-0.64406	0.644069	0.803662	1.223752	1.570668	2.221426
800	-2.22058	-1.57007	-1.22328	-0.80335	-0.64382	0.643825	0.803359	1.223289	1.570074	2.220586
1000	-2.22030	-1.56987	-1.22313	-0.80325	-0.64374	0.643744	0.803258	1.223135	1.569877	2.220307

Table 5: Quantile points of the distribution of Q (when $\rho_{ij} = 0.4$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-2.27934	-1.61162	-1.25565	-0.82461	-0.66086	0.660862	0.824617	1.255659	1.611621	2.279346
20	-2.19394	-1.55123	-1.20861	-0.79372	-0.63610	0.636102	0.793721	1.208614	1.551239	2.193947
30	-2.17125	-1.53519	-1.19611	-0.78551	-0.62952	0.629523	0.785513	1.196115	1.535197	2.171258
40	-2.16075	-1.52777	-1.19032	-0.78171	-0.62647	0.626478	0.781712	1.190328	1.527770	2.160754
50	-2.15469	-1.52348	-1.18699	-0.77952	-0.62472	0.624721	0.779521	1.186991	1.523486	2.154696
60	-2.15075	-1.52070	-1.18482	-0.77809	-0.62357	0.623579	0.778095	1.184820	1.520700	2.150755
70	-2.14798	-1.51874	-1.18329	-0.77709	-0.62277	0.622776	0.777093	1.183294	1.518742	2.147986
80	-2.14593	-1.51729	-1.18216	-0.77635	-0.62218	0.622181	0.776351	1.182164	1.517291	2.145933
90	-2.14435	-1.51617	-1.18129	-0.77577	-0.62172	0.621722	0.775779	1.181292	1.516172	2.144352
100	-2.14309	-1.51528	-1.18060	-0.77532	-0.62135	0.621358	0.775324	1.180600	1.515284	2.143095
200	-2.13753	-1.51135	-1.17753	-0.77331	-0.61974	0.619747	0.773313	1.177539	1.511355	2.137538
300	-2.13571	-1.51006	-1.17653	-0.77265	-0.61921	0.619219	0.772656	1.176537	1.510069	2.135719
400	-2.13481	-1.50943	-1.17604	-0.77232	-0.61895	0.618958	0.772329	1.176040	1.509431	2.134817
500	-2.13427	-1.50904	-1.17574	-0.77213	-0.61880	0.618801	0.772134	1.175742	1.509049	2.134277
800	-2.13347	-1.50847	-1.17529	-0.77184	-0.61856	0.618567	0.771842	1.175298	1.508479	2.133470
1000	-2.13320	-1.50828	-1.17515	-0.77174	-0.61849	0.618490	0.771745	1.175150	1.508289	2.133202

Table 6: Quantile points of the distribution of Q (when $\rho_{ij} = 0.5$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-2.15378	-1.52283	-1.18648	-0.77918	-0.62445	0.624456	0.779189	1.186486	1.522839	2.153780
20	-2.07308	-1.46578	-1.14203	-0.74999	-0.60105	0.601059	0.749996	1.142033	1.465783	2.073085
30	-2.05164	-1.45062	-1.13022	-0.74224	-0.59484	0.594844	0.742240	1.130222	1.450625	2.051646
40	-2.04172	-1.44360	-1.12475	-0.73864	-0.59196	0.591966	0.738649	1.124754	1.443606	2.041720
50	-2.03599	-1.43955	-1.12160	-0.73657	-0.59030	0.590306	0.736578	1.121601	1.439559	2.035996
60	-2.03227	-1.43692	-1.11954	-0.73523	-0.58922	0.589226	0.735231	1.119549	1.436926	2.032272
70	-2.02965	-1.43507	-1.11810	-0.73428	-0.58846	0.588468	0.734284	1.118108	1.435076	2.029656
80	-2.02771	-1.43370	-1.11704	-0.73358	-0.58790	0.587906	0.733583	1.117040	1.433705	2.027716
90	-2.02622	-1.43264	-1.11621	-0.73304	-0.58747	0.587472	0.733042	1.116216	1.432648	2.026222
100	-2.02503	-1.43180	-1.115562	-0.73261	-0.58712	0.587128	0.732612	1.115562	1.431809	2.025035
200	-2.01978	-1.42809	-1.112670	-0.73071	-0.58560	0.585605	0.730712	1.112670	1.428096	2.019783
300	-2.01806	-1.42688	-1.111723	-0.73009	-0.58510	0.585107	0.730091	1.111723	1.426881	2.018065
400	-2.01721	-1.42627	-1.111253	-0.72978	-0.58486	0.584860	0.729782	1.111253	1.426278	2.017212
500	-2.01670	-1.42591	-1.110972	-0.72959	-0.58471	0.584712	0.729598	1.110972	1.425917	2.016702
800	-2.01594	-1.42537	-1.110552	-0.72932	-0.58449	0.584491	0.729322	1.110552	1.425378	2.015940
1000	-2.01568	-1.42519	-1.110413	-0.72923	-0.58442	0.584418	0.729230	1.110413	1.425199	2.015686

Table 7: Quantile points of the distribution of Q (when $\rho_{ij} = 0.6$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-1.98957	-1.40673	-1.09602	-0.71978	-0.57684	0.576847	0.719784	1.096029	1.406738	1.989576
20	-1.91503	-1.35403	-1.05496	-0.69281	-0.55523	0.555235	0.692816	1.054965	1.354032	1.915034
30	-1.89522	-1.34003	-1.04405	-0.68565	-0.54949	0.549493	0.685652	1.044055	1.340030	1.895229
40	-1.88606	-1.33354	-1.04405	-0.68233	-0.54683	0.546835	0.682334	1.044055	1.333547	1.886060
50	-1.88077	-1.32980	-1.03900	-0.68042	-0.54530	0.545301	0.680422	1.039003	1.329808	1.880773
60	-1.87733	-1.32737	-1.03609	-0.67917	-0.54430	0.544304	0.679177	1.036091	1.327376	1.877333
70	-1.87491	-1.32566	-1.03419	-0.67830	-0.54360	0.543603	0.678303	1.034195	1.325667	1.874916
80	-1.87312	-1.32440	-1.03286	-0.67765	-0.54308	0.543084	0.677654	1.032864	1.324400	1.873124
90	-1.87174	-1.32342	-1.03187	-0.67715	-0.54268	0.542684	0.677155	1.031877	1.323424	1.871744
100	-1.87064	-1.32264	-1.031117	-0.67675	-0.54236	0.542366	0.676758	1.031117	1.322648	1.870647
200	-1.86579	-1.31921	-1.03051	-0.67500	-0.54095	0.540959	0.675003	1.030512	1.319218	1.865796
300	-1.86420	-1.31809	-1.02784	-0.67442	-0.54049	0.540499	0.674429	1.027840	1.318096	1.864209
400	-1.86342	-1.31753	-1.02696	-0.67414	-0.54027	0.540271	0.674144	1.026966	1.317539	1.863421
500	-1.86295	-1.31720	-1.02653	-0.67397	-0.54013	0.540134	0.673974	1.026532	1.317206	1.862950
800	-1.86224	-1.31670	-1.02627	-0.67371	-0.53993	0.539930	0.673719	1.026272	1.316708	1.862245
1000	-1.86201	-1.31654	-1.02588	-0.67363	-0.53986	0.539862	0.673634	1.025884	1.316543	1.862011

Table 8: Quantile points of the distribution of Q (when $\rho_{ij} = 0.7$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-1.77605	-1.25576	-0.97840	-0.64253	-0.51493	0.514939	0.642536	0.978401	1.255765	1.776052
20	-1.70951	-1.20871	-0.94174	-0.61846	-0.49564	0.495646	0.618462	0.941744	1.208716	1.709510
30	-1.69183	-1.19621	-0.93200	-0.61206	-0.49052	0.490521	0.612066	0.932005	1.196216	1.691831
40	-1.68364	-1.19042	-0.92749	-0.60910	-0.48814	0.488147	0.609105	0.927496	1.190428	1.683646
50	-1.67892	-1.18709	-0.92489	-0.60739	-0.48677	0.486779	0.607398	0.924895	1.187091	1.678926
60	-1.67585	-1.18492	-0.92320	-0.60628	-0.48588	0.485888	0.606287	0.923204	1.184920	1.675855
70	-1.67369	-1.18339	-0.92201	-0.60550	-0.48526	0.485263	0.605506	0.922015	1.183394	1.673697
80	-1.67209	-1.18226	-0.92113	-0.60492	-0.48479	0.484799	0.604928	0.921134	1.182264	1.672098
90	-1.67086	-1.18139	-0.92045	-0.60448	-0.48444	0.484442	0.604482	0.920455	1.181392	1.670865
100	-1.66988	-1.18070	-0.91991	-0.60412	-0.48415	0.484158	.604128	0.919916	1.180700	1.669886
200	-1.66555	1.17763	-0.91753	-0.60256	-0.48290	0.482903	0.602561	0.917530	1.177638	1.665556
300	-1.66413	-1.17663	-0.91675	-0.60204	-0.48249	0.482492	0.602048	0.916750	1.176636	1.664139
400	-1.66343	-1.17613	-0.91636	-0.60179	-0.48228	0.482288	0.601794	0.916362	1.176139	1.663436
500	-1.66301	-1.17584	-0.91613	-0.60164	-0.48216	0.482166	0.601642	0.916131	1.175842	1.663015
800	-1.66238	-1.17539	-0.91578	-0.60141	-0.48198	0.481984	0.601414	0.915784	1.175397	1.662387
1000	-1.66217	-1.17524	-0.91566	-0.60133	-0.48192	0.481923	0.601339	0.915669	1.175249	1.662178

Table 9: Quantile points of the distribution of Q (when $\rho_{ij} == 0.8$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-1.49218	-1.05505	-0.82202	-0.53983	-0.43263	0.432635	0.539838	0.822021	1.055053	1.492182
20	-1.43627	-1.01552	-0.79122	-0.51961	-0.41642	0.416426	0.519612	0.791223	1.015524	1.436275
30	-1.42142	-1.00502	-0.78304	-0.51423	-0.41212	0.412120	0.514239	0.783040	1.005022	1.421422
40	-1.41454	-1.00015	-0.77925	-0.51175	-0.41012	0.410126	0.511751	0.779252	1.000159	1.414545
50	-1.41058	-0.99735	-0.77706	-0.51031	-0.40897	0.408976	0.510316	0.777068	0.997356	1.410580
60	-1.40799	-0.99553	-0.77564	-0.50938	-0.40822	0.408228	0.509383	0.775646	0.995531	1.407999
70	-1.40618	-0.99424	-0.77464	-0.50872	-0.40770	0.407702	0.508727	0.774648	0.994249	1.406187
80	-1.40484	-0.99330	-0.77390	-0.50824	-0.40731	0.407313	0.508241	0.773907	0.993300	1.404843
90	-1.40380	-0.99256	-0.77333	-0.50786	-0.40701	0.407013	0.507866	0.773337	0.992567	1.403808
100	-1.40298	-0.99198	-0.77288	-0.50756	-0.40677	0.406774	0.507569	0.772884	0.991986	1.402985
200	-1.39934	-0.98941	-0.77088	-0.50625	-0.40571	0.405719	0.506252	0.770880	0.989413	1.399347
300	-1.39815	-0.98857	-0.77022	-0.50582	-0.40537	0.405374	0.505822	0.770224	0.988572	1.398157
400	-1.39756	-0.98815	-0.76989	-0.50560	-0.40520	0.405203	0.505608	0.769898	0.988154	1.397566
500	-1.39721	-0.98790	-0.76970	-0.50548	-0.40510	0.405100	0.505480	0.769704	0.987904	1.397212
800	-1.39668	-0.98753	-0.76941	-0.50528	-0.40494	0.404947	0.505289	0.769413	0.987531	1.396684
1000	-1.39650	-0.98740	-0.76931	-0.50522	-0.40489	0.404896	0.505225	0.769316	0.987407	1.396509

Table 10: Quantile points of the distribution of Q (when $\rho_{ij} = 0.9$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-1.08404	-0.76647	-0.59718	-0.39218	-0.31430	0.314302	0.392183	0.597185	0.766478	1.084045
20	-1.04343	-0.73776	-0.57481	-0.37748	-0.30252	0.302526	0.377489	0.574810	0.737761	1.043430
30	-1.03263	-0.73013	-0.56886	-0.37358	-0.29939	0.299398	0.373586	0.568866	0.730131	1.032639
40	-1.02764	-0.72659	-0.56611	-0.37177	-0.29794	0.297949	0.371778	0.566113	0.726599	1.027643
50	-1.02476	-0.72456	-0.56452	-0.37073	-0.29711	0.297114	0.370736	0.564526	0.724562	1.024762
60	-1.02288	-0.72323	-0.56349	-0.37005	-0.29657	0.296571	0.370058	0.563494	0.723237	1.022888
70	-1.02157	-0.72230	-0.56276	-0.36958	-0.29618	0.296189	0.369581	0.562768	0.722305	1.021571
80	-1.02059	-0.72161	-0.56223	-0.36922	-0.29590	0.295906	0.369228	0.562231	0.721615	1.020595
90	-1.01984	-0.72108	-0.56181	-0.36895	-0.29568	0.295688	0.368956	0.561816	0.721083	1.019843
100	-1.01924	-0.72066	-0.56148	-0.36874	-0.29551	0.295514	0.368740	0.561487	0.720661	1.019245
200	-1.01660	-0.71879	-0.56003	-0.36778	-0.29474	0.294748	0.367784	0.560031	0.718792	1.016602
300	-1.01573	-0.71818	-0.55955	-0.36747	-0.29449	0.294497	0.367471	0.559555	0.718181	1.015737
400	-1.01530	-0.71787	-0.55931	-0.36731	-0.29437	0.294373	0.367315	0.559318	0.717877	1.015308
500	-1.01505	-0.71769	-0.55917	-0.36722	-0.29429	0.294298	0.367223	0.559177	0.717696	1.015051
800	-1.01466	-0.71742	-0.55896	-0.36708	-0.29418	0.294187	0.367084	0.558965	0.717424	1.014667
1000	-1.01454	-0.71733	-0.55889	-0.36703	-0.29415	0.294150	0.367038	0.558895	0.717334	1.014540

Table 11: Quantile points of the distribution of Q (when $\rho_{ij} = 0.99$)

n	$Q_{0.01}$	$Q_{0.05}$	$Q_{0.10}$	$Q_{0.20}$	$Q_{0.25}$	$Q_{0.75}$	$Q_{0.80}$	$Q_{0.90}$	$Q_{0.95}$	$Q_{0.99}$
10	-0.35083	-0.24805	-0.19326	-0.12692	-0.10171	-0.35083	-0.24805	-0.19326	-0.12692	-0.10171
20	-0.33768	-0.23876	-0.18602	-0.12216	-0.09790	-0.33768	-0.23876	-0.18602	-0.12216	-0.09790
30	-0.33419	-0.23629	-0.18410	-0.12090	-0.09689	-0.33419	-0.23629	-0.18410	-0.12090	-0.09689
40	-0.33257	-0.23514	-0.18321	-0.12031	-0.09642	-0.33257	-0.23514	-0.18321	-0.12031	-0.09642
50	-0.33164	-0.23449	-0.18269	-0.11998	-0.09615	-0.33164	-0.23449	-0.18269	-0.11998	-0.09615
60	-0.33103	-0.23406	-0.18236	-0.11976	-0.09597	-0.33103	-0.23406	-0.18236	-0.11976	-0.09597
70	-0.33061	-0.23376	-0.18212	-0.11960	-0.09585	-0.33061	-0.23376	-0.18212	-0.11960	-0.09585
80	-0.33029	-0.23353	-0.18195	-0.11949	-0.09576	-0.33029	-0.23353	-0.18195	-0.11949	-0.09576
90	-0.33005	-0.23336	-0.18182	-0.11940	-0.09569	-0.33005	-0.23336	-0.18182	-0.11940	-0.09569
100	-0.32985	-0.23322	-0.18171	-0.11933	-0.09563	-0.32985	-0.23322	-0.18171	-0.11933	-0.09563
200	-0.32900	-0.23262	-0.18124	-0.11902	-0.09538	-0.32900	-0.23262	-0.18124	-0.11902	-0.09538
300	-0.32872	-0.23242	-0.18108	-0.11892	-0.09530	-0.32872	-0.23242	-0.18108	-0.11892	-0.09530
400	-0.32858	-0.23232	-0.18101	-0.11887	-0.09526	-0.32858	-0.23232	-0.18101	-0.11887	-0.09526
500	-0.32850	-0.23226	-0.18096	-0.11884	-0.09524	-0.32850	-0.23226	-0.18096	-0.11884	-0.09524
800	-0.32837	-0.23218	-0.18089	-0.11879	-0.09520	-0.32837	-0.23218	-0.18089	-0.11879	-0.09520
1000	-0.32833	-0.23215	-0.18087	-0.11878	-0.09519	-0.32833	-0.23215	-0.18087	-0.11878	-0.09519