Parameter Estimation of the Hybrid Censored Lomax Distribution

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Abstract

Survival analysis is used in various fields for analyzing data involving the duration between two events. It is also known as event history analysis, lifetime data analysis, reliability analysis or time to event analysis. One of the difficulties which arise in this area is the presence of censored data. The lifetime of an individual is censored when it cannot be exactly measured but partial information is available. Different circumstances can produce different types of censoring. The two most common censoring schemes used in life testing experiments are Type-I and Type-II censoring schemes. Hybrid censoring scheme is mixture of Type-I and Type-II censoring scheme. In this paper we consider the estimation of parameters of Lomax distribution based on hybrid censored data. The parameters are estimated by the maximum likelihood and Bayesian methods. The Fisher information matrix has been obtained and it can be used for constructing asymptotic confidence intervals.

Keywords: Lomax distribution; Maximum likelihood estimators; Bayesian inferences; Asymptotic Variance covariance matrix; Type-I censoring; Type-II censoring; Hybrid censoring scheme.

1. Introduction

The Lomax distribution was originally proposed as a second kind of the Pareto distribution by Lomax (1954). It is used to provide a good model in biomedical problems. It is considered as an important model of lifetime models. Also, it has been used in relation with studies of income, size of cities and reliability modeling. It is being widely used for stochastic modeling of decreasing failure rate life components. It also serves as a useful model in the study of queuing theory and biological analysis.

Type-I and Type-II censoring schemes are the two most popular censoring schemes which are used in practice. They can be briefly described as follows: Suppose n units are put on a life test. In Type-I censoring, experiment continues up to a pre-specified time T. Failures after the time T are not observed. In Type-II censoring, the experiment continues till the r-th failure takes place, where r is a pre-specified integer and $r \leq n$. Therefore, in Type-I censoring scheme, the number of failures is random and in Type-II censoring scheme, the experimental time is random. A mixture of Type-I and Type-II schemes is known as the hybrid censoring scheme. The hybrid censoring scheme was first introduced by Epstein (1954, 1960). But recently it becomes quite popular in the reliability and life-testing experiments. Suppose n identical units are put on a life test. The test is terminated when a pre-specified number r, out of n units have failed or a pre-determined time T, has been reached. Therefore, in hybrid censoring scheme, the experimental time and the number of failures will not exceed T and r respectively. It is clear that Type-I and Type-II censoring schemes can be obtained as special cases of hybrid censoring scheme by taking $r = n$ and $T = \infty$ respectively.

Epstein (1954) first introduced the hybrid censoring scheme and analyzed the data under the assumption of exponential lifetime distribution of the experimental units. He also proposed a two sided confidence interval of the unknown parameter, without any formal proof. Fairbanks et al. (1982) modified slightly the proposition of Epstein (1954) and suggested a simple set of confidence intervals. Chen and Bhattacharya (1988) obtained the exact one sided confidence interval based on the distribution of the maximum likelihood estimator of the exponential parameter. Drapper and Guttmann (1987) also considered the same problem but from the Bayesian point of view, and obtained two sided credible interval of the mean lifetime using the inverted gamma prior. Comparisons and criticisms of the different methods can be found in Gupta and Kundu (1998). For some of the relevant references on hybrid censoring and related topics the readers are referred to Ebrahimi (1990,1992), Jeong et al.(1996), Childs et al. (2003), Kundu (2007), Banerjee and Kundu (2008), Kundu and Pradhan (2009) and Dube, et al. (2010). It may be mentioned that although hybrid censoring scheme seems to be an important censoring scheme, but not much work has been done other than the exponential, Weibull or log-normal distributions. In this article we consider the hybrid censored lifetime data, when the lifetime follows two-parameter Lomax distribution. The main aim of this paper is two-fold. First, we provide different methods to compute the point estimations of the unknown parameters. We provide the MLEs of the unknown parameters. It is observed that the MLEs can be obtained by solving a non-linear equation and we will use a simple iterative scheme to solve the non-linear equation. We also discuss the properties of the estimators by obtaining the variance covariance matrix.

The second aim is to provide the Bayes estimates under the assumptions of informative and conjugate priors on the parameters.

It is observed that the Bayes estimates cannot be computed explicitly, and we use MathCAD software to compute the Bayes estimates. We compare the performances of the different methods by Monte Carlo simulations.

2. Maximum Likelihood Estimation

Suppose X is distributed as Lomax distribution with probability density function and the cumulative distribution function defined respectively as follows:

$$f(x)=\frac{\alpha}{\lambda}(1+\frac{x}{\lambda})^{-(\alpha+1)} \qquad x>0,\lambda,\alpha>0 \tag{2.1}$$

$$F(x)=1-(1+\frac{x}{\lambda})^{-\alpha}, \qquad x>0,\lambda,\alpha>0, \tag{2.2}$$

where λ is the scale parameter and α is the shape parameter.
We will use maximum likelihood method to estimate the unknown parameters of Lomax distribution using hybrid censored sample. Suppose n identical units are put on life test. The test is terminated when a pre-chosen number, r, out of n items has failed or when a pre-determined time, T, on test has been reached. It is also assumed that the failed items are not replaced under the hybrid censoring scheme, it is assumed that r and T are known in advance.

The likelihood function for the hybrid censored data may be written as

$$L(\theta)=\frac{n!}{(n-D^*)!}\left(\prod_{i=1}^{D^*} f(x_{(i)},\theta)\right)\left(1-F(T^*,\theta)\right)^{n-D^*} \tag{2.3}$$

T^*=min $(T, x_{(r)})$, and D*: denotes the number of units that would fail before the time T^* see (Kundu (2007)).

Considering the likelihood function (2.3) and the density function (2.1), we have:

$$L(\alpha,\lambda,\beta)\propto\left(\frac{\alpha}{\lambda}\right)^{D*}\left(\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})^{-(\alpha+1)}\right)\left((1+\frac{T^*}{\lambda})^{-\alpha}\right)^{n-D^*} \tag{2.4}$$

the logarithm of (2.4) will be

$$l=InL(\alpha,\lambda)\propto D^* In\alpha-D^* In\lambda-(\alpha+1)\sum_{i=1}^{D^*} In(1+\frac{x_{(i)}}{\lambda})-\alpha(n-D^*)In(1+\frac{T^*}{\lambda}) \tag{2.5}$$

Differentiating (2.5) with respect to α and λ, and equality to zero, we obtain the following likelihood equations:

$$\frac{D^*}{\hat{\alpha}}-\sum_{i=1}^{D^*} In(1+\frac{x_{(i)}}{\hat{\lambda}})+(n-D^*)In(1+\frac{T^*}{\hat{\lambda}})=0 \tag{2.6}$$

and

$$\frac{D^*}{\hat{\lambda}} - (\alpha+1)\sum_{i=1}^{D^*}\left(\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}\right) + \alpha(n-D^*)\frac{T^*}{\hat{\lambda}(\hat{\lambda}+x_{(i)})} = 0 \qquad (2.7)$$

From (2.6) we have

$$\frac{D^*}{\hat{\alpha}} = \sum_{i=1}^{D^*}\ln\left(1+\frac{x_{(i)}}{\hat{\lambda}}\right) + (n-k)\ln\left(1+\frac{T^*}{\hat{\lambda}}\right)$$

then

$$\hat{\alpha} = \frac{D^*}{\sum_{i=1}^{D^*}\ln\left(1+\frac{x_{(i)}}{\hat{\lambda}}\right) - (n-D^*)\ln\left(1+\frac{T^*}{\hat{\lambda}}\right)} = \mathrm{U}(\hat{\lambda}) \qquad (2.8)$$

Using (2.8), (2.7) can be written as

$$\frac{D^*}{\hat{\lambda}} = (u(\hat{\lambda})+1)\sum_{i=1}^{D^*}\left(\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}\right) + u(\hat{\lambda})(n-D^*)\left(\frac{T^*}{\hat{\lambda}(\hat{\lambda}+T)}\right)$$

then

$$\hat{\lambda} = \frac{D^*}{(u(\hat{\lambda})+1)\sum_{i=1}^{D^*}\left(\frac{x_{(i)}}{\hat{\lambda}(\hat{\lambda}+x_{(i)})}\right) + u(\hat{\lambda})(n-D^*)\left(\frac{T^*}{\hat{\lambda}(\hat{\lambda}+T^*)}\right)} \qquad (2.9)$$

Clearly computer facilities and numerical method are needed to obtain $\hat{\alpha}$ and $\hat{\lambda}$.

The asymptotic variance-covariance matrix of ($\hat{\alpha}$, $\hat{\lambda}$) is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates [see Cohen (1965)]. Accordingly; we have the following approximate variance-covariance matrix

$$\hat{I}_1(\hat{\alpha},\hat{\lambda}) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\lambda\partial\alpha} \\ \frac{\partial^2 \ln L}{\partial\lambda\partial\alpha} & \frac{\partial^2 \ln L}{\partial\lambda^2} \end{bmatrix}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}} \qquad (2.10)$$

The element of Fisher information matrix are given as follows:

$$-\left.\frac{\partial^2 l}{\partial\alpha^2}\right|_{\hat{\alpha},\hat{\lambda}} = \frac{-D^*}{\hat{\alpha}^2}$$

$$-\frac{\partial^2 l}{\partial \lambda^2}\bigg|_{\hat{\alpha},\hat{\lambda}} = \frac{D^*}{\hat{\lambda}^2} - \sum_{i=1}^{n}(1+\frac{(x_{(i)})(2\hat{\lambda}-x_{(i)})}{\hat{\lambda}^2(\hat{\lambda}+x_{(i)})^2}) + \hat{\alpha}(n-D^*)\frac{(T^*)(2\hat{\lambda}-T^*)}{\hat{\lambda}^2(\hat{\lambda}+T^*)^2}$$

and

$$-\frac{\partial^2 l}{\partial \alpha \partial \lambda}\bigg|_{\hat{\alpha},\hat{\lambda}} = -\frac{\partial^2 l}{\partial \lambda \partial \alpha}\bigg|_{\hat{\alpha},\hat{\lambda}} = \sum_{i=1}^{n}(\frac{x_{(i)}}{\hat{\lambda}[\hat{\lambda}+x_{(i)}]}) + (n-D^*)\frac{T^*}{\hat{\lambda}[\hat{\lambda}+T^*]}$$

3. Bayesian Inferences

To obtain the Bayes estimations of the unknown parameters for two parameter Lomax distribution under hybrid censored samples, we shall use the likelihood function (2.4), assume that the two-parameter α and λ are independent and let the non-informative prior (NIP), the function for α and λ are respectively given by:

$$\pi(\alpha) \propto \alpha^{k-1} \qquad \alpha > 0, \quad k > 0$$

and

$$\pi(\lambda) \propto \lambda^{-1} \qquad \lambda > 0, \quad b > 0$$

Hence, the joint prior density of α and λ will be

$$\pi(\alpha,\lambda) \propto \alpha^{k-1}\lambda^{-1} \qquad \alpha > 0, \qquad \lambda > 0 \tag{3.1}$$

Based on the (2.4) and (3.1), the joint posterior density functions of α and λ under hybrid censored sample will be:

$$f(\alpha,\lambda|x) = \frac{\lambda^{D^*-1}\alpha^{D^*+k-1}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})}{\psi}. \qquad \alpha > 0, \quad \lambda > 0$$

Where, ψ is normalized constant equal to

$$\psi = \int_0^\infty\int_0^\infty \lambda^{D^*-1}\alpha^{D^*+k-1}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})d\alpha d\lambda$$

The joint mode of the bivariate posterior distribution. (3.1) may be considered as Bays' estimates. clearly if $K=1$ and $\lambda=1$, maximum likelihood estimates will be the same as the joint posterior mode.

Now, the marginal posterior of any parameter is obtained by integrating the joint posterior distribution with respect to the other parameters, the posterior probability density function of α can be obtained:

$$f(\alpha|x)=\frac{\alpha^{D^*+k-1}\int_0^\infty \lambda^{D^*-1}.\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})\quad d\lambda}{\psi}$$

Similarly integrating the joint posterior (3.1) with respect to λ, the marginal posterior of λ can be obtained as:

$$f(\lambda|x)=\frac{\lambda^{D^*-1}\int_0^\infty \alpha^{D^*+k-1}.\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})\quad d\alpha}{\psi}$$

It is well known that under a squared error loss function, the Bayes estimator of a parameter will be its posterior expectation and posterior variance will be the Bayes' risk. To obtain the posterior mean and variance a numerical integration is required. Then, the posterior mean and variance of the shape and scale parameter (α,λ) are expressed as follows:

$$E(\alpha|\lambda,x)=\tilde{\alpha}=\int_0^\infty\int_0^\infty \lambda^{D^*-1}\alpha^{D^*+k}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})\quad d\lambda d\alpha\quad, \tag{3.2}$$

$$\operatorname{var}(\alpha|\lambda,x)=E(\tilde{\alpha}-\alpha)^2$$

$$\operatorname{var}(\alpha|\lambda,x)=\int_0^\infty\int_0^\infty(\tilde{\alpha}-\alpha)^2\lambda^{D^*-1}\alpha^{D^*+k-1}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})d\lambda d\alpha \tag{3.3}$$

Similarly

$$E(\lambda|\alpha,x)=\tilde{\lambda}=\int_0^\infty\int_0^\infty \lambda\lambda^{D^*-1}\alpha^{D^*+k-1}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})\quad d\alpha d\lambda\quad, \tag{3.4}$$

$$\operatorname{var}(\lambda|\alpha,x)=E(\tilde{\lambda}-\lambda)^2$$

$$\operatorname{var}(\lambda|\alpha,x)=\int_0^\infty\int_0^\infty(\tilde{\lambda}-\lambda)^2\lambda^{D^*-1}\alpha^{D^*+k-1}\prod_{i=1}^{D^*}(1+\frac{x_{(i)}}{\lambda})(1+\frac{T^*}{\lambda})\quad d\alpha d\lambda \tag{3.5}$$

Equation (3.2) to (3.5) are very hard to evaluate theoretically obtain, a numerical procedure is needed to solve these equations numerically, MathCAD 2001 package will be used to obtain the posterior mean and variance of the shape and scale parameter (α,λ).

4. Numerical Results

In this section we carry out a simulation study to compare the performances of the MLEs and the Bayesian. The simulation is carried out for different choices of n, r and T values. For a particular set of hybrid censored data, the MLEs and Bayesian are obtained as described before. It is clear that, there are no explicit solutions for obtaining new estimators either in both non-Bayesian and Bayesian approaches. Therefore random data, numerical solution and computer facilities are needed. The main object of this section is to illustrate numerically most of the new theoretical result obtained in the previous subsections.

Using MathCAD 2001 package, the maximum likelihood estimators and their variance covariance matrix for the unknown parameters of Lomax distribution under hybrid censored sample will be obtained according to the following steps:
Step 1: random sample of size 10, 15, 20, 25, 30 and 50 were generated from Lomax distribution. The generation of random nimbus from the Lomax distribution is very simple, if U has a uniform (0, 1) random number, then

$$X = \lambda\left[\sqrt[\alpha]{\frac{1}{1-U}} - 1\right]$$

follows a Lomax distribution. The true parameters selected values are α =(1,2,3), λ =(1.2,2,2.5).

Step 2: In hybrid censored sample, choose the censoring time T and failed r.

Step 3: Solving the non linear likelihood for $\hat{\alpha}$ and $\hat{\lambda}$ in equations (2.8) and (2.9), respectively, to obtain the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\lambda}$.

Step 4: using equation (2.10) to obtain the variance covariance matrix of ($\hat{\alpha}, \hat{\lambda}$). Results tabulated in Table 1 (see appendix).

From Table 1, it is noted that the standard deviation decreases when n is increasing, when α and λ at (1,2,3) and (1.2,2,2.5) respectively, similarly the maximum likelihood estimator of the parameters has the same behaviors when the sample size becomes large and the properties of two parameters α and λ at (1 and 1.2) respectively is better than the anther values.

To obtain the posterior mean and the posterior variance of α, λ, the numerical procedures will describe as follow:
Repeat step 1 and 2 above,

Step 3: In hybrid censored sample, choose the censoring time T, failed r and prior k.

Step 4: Solving the non linear Bayesian for posterior variance of the shape and scale parameter (α, λ).in equations (3.2) to (3.5).

Step 5: the posterior mean and the posterior variance of the estimators for the shape and scale parameter (α, λ) for all sample size and for sets of parameters were obtained.

Numerical results are summarized in Table 2 (see appendix), it is noted that the posterior mean decreases when n is increasing. Similarly the posterior variance of the parameters has the same behaviors when the sample size becomes large.

References

1. Banerjee, A., Kundu, D., 2008. Inference based on Type-II hybrid censored data from a Weibull distribution. IEEE Transactions on Reliability 57, 369-378.
2. Chen, S., Bhattacharya, G.K., 1988. Exact confidence bounds for an exponential parameter under hybrid censoring. Communications in Statistics - Theory and Methods 17, 1857-1870.
3. Childs, A., Chandrasekhar, B., Balakrishnan, N., Kundu, D., 2003. Exact inference based on type-I and type-II hybrid censored samples from the exponential distribution. Annals of the Institute of Statistical Mathematics 55, 319-330.
4. Cohen, A.C., (1965). Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and Censored Samples, Technometrics, 7: 579-588.
5. Draper, N., Guttman, I., 1987. Bayesian analysis of hybrid life tests with exponential failure times. Annals of the Institute of Statistical Mathematics 39, 219- 225.
6. Ebrahimi, N., 1990. Estimating the parameter of an exponential distribution from hybrid life test. Journal of Statistical Planning and Inference 23, 255 - 261.
7. Ebrahimi, N., 1992. Prediction intervals for future failures in exponential distribution under hybrid censoring. IEEE Transactions on Reliability 41, 127 - 132.
8. Epstein, B., 1954. Truncated life tests in the exponential case. Annals of Mathematical Statistics 25, 555 - 564.
9. Epstein, B. 1960, Estimation from life-test data, Technometrics, Vol. 2, 447 - 454.
10. Fairbanks, K., Madson, R., Dykstra, R., 1982. A confidence interval for an exponential parameter from a hybrid life test. Journal of the American Statistical Association 77, 137 - 140.
11. Gupta, R.D., Kundu, D., 1998. Hybrid censoring schemes with exponential failure distribution. Communications in Statistics - Theory and Methods 27, 3065 - 3083.

12. Jeong, H.S., Park, J.I., Yum, B.J., 1996. Development of (r, T) hybrid sampling plans for exponential lifetime distributions. Journal of Applied Statistics 23, 601 - 607.

13. Kundu, D., 2007. On hybrid censored Weibull distribution. Journal of Statistical Planning and Inference 137, 2127 - 2142.

14. Kundu, D., Pradhan, B. 2009. Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring. Communications in Statistics Theory and Methods 38, 2030 - 2041.

15. Lomax, K.S., 1954. Business failures. Another example of the analysis of failure data. Journal of the American Statistical Association, 49: 847-852.

16. Dube, S., Pradhan, B. and Kundu, D. (2010). Parameter estimation of the hybrid censored log-normal distribution. To appear at Journal of Statistical Computation and Simulation.

Appendix

Table 1: The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 1, \quad \lambda = 1.2$)

n	r	T	H	$\widehat{\alpha}$		$\widehat{\lambda}$		$Co\,\text{var}$
				MLE	Standard deviation	MLE	Standard deviation	$(\widehat{\alpha},\widehat{\lambda})$
	6		0	1.597	0.597	1.737	0.537	30.241
		0.8	0	0.558	0.442	1.649	0.449	28.763
10	8		0	0.571	0.429	1.448	0.248	27.22
			1	1.534	0.534	1.767	0.567	15.743
	10	1.5	0	1.402	0.402	1.544	0.344	15.443
			0	1.068	0.068	1.487	0.287	15.317
			0	1.525	0.525	1.583	0.383	16.975
	7	0.8	0	1.44	0.44	1.439	0.239	16.418
15	9		0	0.739	0.261	1.216	0.016	15.579
			1	1.187	0.187	1.591	0.391	15.339
	15	1.5	1	1.084	0.084	1.542	0.342	15.186
			0	1.003	3.053×10^{-3}	1.332	0.132	14.741
			0	1.23	0.23	1.662	0.462	11.004
	18	0.8	0	1.937	0.063	1.618	0.418	10.931
25	22		0	0.943	0.057	1.578	0.378	10.011
			1	1.475	0.475	1.45	0.25	9.061
	25	1.5	1	0.686	0.314	1.354	0.154	8.444
			1	0.872	0.128	1.267	0.067	8.383
	9		1	0.733	0.267	1.49	0.29	8.869
30		0.8	0	0.865	0.135	1.414	0.214	8.792
	25		0	1.134	0.134	1.355	0.155	8.245

	30	1.5	1 0 0	0.426 1.372 0.819	0.574 0.372 0.181	1.719 1.548 1.209	0.519 0.348 9.0×10^{-3}	5.19 4.851 4.722
50	25 30	0.8	0 0 0	0.447 0.474 1.218	0.553 0.526 0.218	1.39 1.317 1.234	0.19 0.117 0.034	1.987 1.854 1.823
	50	1.5	1 1 0	0.55 1.129 1.035	0.45 0.129 0.035	1.702 1.561 1.385	0.502 0.361 0.185	1.729 1.432 1.408

H : terminate the experiment at min ($T, x_{(r)}$).

$$H = \begin{vmatrix} 0 & if & H = T \\ 1 & if & H = x_{(r)} \end{vmatrix}$$

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 1, \quad \lambda = 2$)

n	r	T	H	$\hat{\alpha}$		$\hat{\lambda}$		$Co\,\text{var}$ $(\hat{\alpha}, \hat{\lambda})$
				MLE	Standard deviation	MLE	Standard deviation	
10	5 9	0.8	0 0 1	1.395 1.501 1.014	0.395 0.501 0.014	2.717 2.585 2.461	0.717 0.585 0.461	42.175 35.214 18.258
	10	1.5	1 1 0	1.284 1.481 1.008	0.284 0.481 0.008	2.648 2.441 2.314	0.648 0.441 0.314	40.954 33.254 16.418
15	8 12	0.8	0 0 0	2.982 1.921 1.604	1.982 0.921 0.604	3.194 2.971 2.641	1.194 0.971 0.641	48.281 33.621 27.214
	15	1.5	1 0 0	2.421 1.748 1.554	1.421 0.748 0.554	3.019 2.684 2.458	1.019 0.684 0.458	40.218 31.257 22.147
25	7 13	0.8	0 0 0	1.541 1.024 1.07	0.541 0.024 0.07	3.141 2.211 1.712	1.141 0.211 0.288	70.219 40.596 45.214
	25	1.5	1 1 1	1.658 1.018 1.001	0.658 0.018 0.001	2.995 2.147 1.654	0.995 0.147 0.346	68.284 50.258 52.547
30	9 27	0.8	1 1 0	1.374 1.173 1.017	0.374 0.173 0.017	3.183 2.808 2.417	1.183 0.808 0.417	77.328 60.982 58.785

	30	1.5	1	1.228	0.228	2.954	0.954	68.952
			0	1.109	0.109	2.756	0.756	53.218
			0	1.019	0.019	2.341	0.341	40.021
50	34	0.8	0	1.721	0.721	2.481	0.481	77.248
			0	1.856	0.856	2.441	0.441	80.514
	47		1	1.007	0.007	1.761	0.239	54.951
		1.5	1	1.652	0.652	2.324	0.324	56.254
	50		1	1.421	0.421	2.228	0.228	48.954
			0	1.004	0.004	1.654	0.346	51.217

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when

$(\alpha = 1, \quad \lambda = 2.5)$

n	r	T	H	$\hat{\alpha}$		$\hat{\lambda}$		$Co\,\text{var}$ $(\hat{\alpha},\hat{\lambda})$
				MLE	Standard deviation	MLE	Standard deviation	
10	3	0.8	0	1.708	0.708	6.019	3.519	40.258
			0	1.521	0.521	6.241	3.741	36.284
	9		0	1.248	0.248	5.887	3.387	20.579
		1.5	1	1.689	0.689	5.987	3.487	37.325
	10		0	1.449	0.449	5.675	3.175	33.147
			0	1.189	0.189	5.324	2.824	18.798
15	8	0.8	0	1.628	0.628	5.518	3.018	50.598
			1	1.504	0.504	5.425	2.925	43.275
	12		1	1.291	0.291	5.184	2.684	30.211
		1.5	1	1.458	0.458	5.324	2.824	46.257
	15		0	1.359	0.359	5.287	2.787	41.145
			0	1.201	0.201	5.014	2.514	28.145
25	7	0.8	0	1.638	0.638	5.521	3.021	64.194
			0	1.507	0.507	5.355	2.855	59.058
	12		1	1.198	0.198	5.007	2.507	28.708
		1.5	1	1.579	0.579	5.47	2.97	60.408
	25		1	1.475	0.475	5.138	2.638	44.787
			0	1.188	0.188	4.684	2.184	23.147
30	3	0.8	1	1.707	0.707	5.471	2.971	72.819
			1	1.718	0.718	5.058	2.558	62.23
	23		0	1.311	0.311	4.845	2.345	39.105
		1.5	0	1.667	0.667	5.147	2.647	64.952
	30		0	1.472	0.472	5.004	2.504	55.982
			0	1.254	0.254	4.685	2.185	31.471

50	14	0.8	0	1.584	0.584	4.559	2.059	78.028
			0	1.428	0.428	4.129	1.629	62.095
	22		1	0.95	0.05	3.968	1.468	64.194
		1.5	1	1.478	0.478	4.328	1.828	71.542
	50		0	1.336	0.336	4.099	1.599	63.254
			0	0.72	0.28	3.856	1.356	44.851

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 2, \quad \lambda = 1.2$)

n	r	T	H	$\hat{\alpha}$		$\hat{\lambda}$		$Co\,\text{var}(\hat{\alpha},\hat{\lambda})$
				MLE	Standard deviation	MLE	Standard deviation	
10	4	0.8	0	2.725	0.725	2.977	1.777	38.204
			0	2.527	0.527	2.527	1.327	27.286
	6		1	2.304	0.304	2.022	0.822	21.479
		1.5	0	2.654	0.654	2.745	1.545	32.148
	10		0	2.447	0.447	2.448	1.248	24.857
			0	2.214	0.214	2.018	0.818	18.147
15	5	0.8	0	2.912	0.912	2.691	1.491	49.217
			0	2.724	0.724	2.397	1.197	34.874
	12		1	2.299	0.299	1.997	0.797	27.258
		1.5	0	2.842	0.842	2.542	1.342	42.578
	15		0	2.564	0.564	2.287	1.087	31.504
			0	2.187	0.187	1.854	0.654	21.074
25	8	0.8	0	2.884	0.884	2.514	1.314	68.955
			1	2.737	0.737	2.067	0.867	51.857
	14		1	2.348	0.348	1.802	0.602	37.079
		1.5	1	2.645	0.645	2.448	1.248	58.217
	25		0	2.514	0.514	2.009	0.809	48.574
			0	2.217	0.217	1.742	0.542	31.579
30	19	0.8	1	2.521	0.521	2.442	1.242	71.048
			1	2.308	0.308	1.995	0.795	50.257
	21		0	2.295	0.295	1.749	0.549	44.275
		1.5	1	2.447	0.447	2.357	1.157	66.985
	30		0	2.189	0.189	1.782	0.582	41.782
			0	2.172	0.172	1.624	0.424	36.284
50	17	0.8	0	2.578	0.578	2.391	1.191	88.086
			0	2.294	0.294	2.012	0.812	52.717
	29		1	2.188	0.188	1.667	0.467	42.549

			1	2.449	0.449	2.174	0.974	68.594
	50	1.5	1	2.184	0.184	1.954	0.754	48.254
			0	2.107	0.107	1.348	0.148	33.854

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when $(\alpha = 2, \quad \lambda = 2)$

n	r	T	H	$\widehat{\alpha}$		$\widehat{\lambda}$		$Co\,\text{var}$ $(\widehat{\alpha}, \widehat{\lambda})$
				MLE	Standard deviation	MLE	Standard deviation	
	3		0	3.791	1.791	4.849	2.849	42.679
		0.8	0	3.631	1.631	4.484	2.484	22.581
			1	3.487	1.487	4.326	2.326	14.109
10	9		1	3.568	1.568	4.625	2.625	35.487
		1.5	1	3.472	1.472	4.532	2.32	19.854
	10		0	3.324	1.324	4.318	2.318	11.524
	7		0	2.931	0.931	4.751	2.751	69.093
		0.8	0	2.584	0.584	3.654	1.654	38.301
			1	2.357	0.357	3.271	1.271	24.187
15	12		1	2.874	0.874	4.895	2.895	62.587
		1.5	0	2.647	0.647	4.724	2.724	33.597
	15		0	2.248	0.248	3.174	1.174	21.571
	18		0	3.761	1.761	4.501	2.501	61.799
		0.8	0	2.401	0.401	3.718	1.718	38.365
			1	2.259	0.259	3.089	1.089	30.009
25	22		1	3.784	1.784	4.658	2.658	51.879
		1.5	1	2.547	0.547	3.647	1.647	43.971
	25		0	2.147	0.147	3.065	1.065	23.941
	20		1	3.201	1.201	4.481	2.481	53.817
		0.8	0	2.482	0.482	3.332	1.332	41.785
			0	2.511	0.511	3.146	1.146	28.997
30	27		1	3.008	1.008	4.418	2.418	49.287
		1.5	1	2.418	0.418	3.217	1.217	32.871
	30		0	2.351	0.351	3.027	1.027	20.254
	11		0	2.841	0.841	4.211	2.211	84.749
		0.8	0	2.318	0.318	3.916	1.916	41.401
			1	2.488	0.488	3.038	1.038	38.815
50	37		0	2.654	0.654	4.845	2.845	72.217
		1.5	0	2.448	0.448	3.851	1.851	33.81
	50		0	2.147	0.147	3.024	1.024	22.048

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 2, \quad \lambda = 2.5$)

n	r	T	H	$\hat{\alpha}$ *MLE*	$\hat{\alpha}$ Standard deviation	$\hat{\lambda}$ *MLE*	$\hat{\lambda}$ Standard deviation	$Co\,\mathrm{var}\,(\hat{\alpha},\hat{\lambda})$
10	3	0.8	0	3.978	1.978	6.843	4.343	44.982
			1	3.754	1.754	6.684	4.184	48.311
	8		1	3.421	1.421	6.199	3.699	24.971
		1.5	1	3.924	1.924	5.987	3.487	34.157
	10		1	3.958	1.958	5.871	3.371	32.574
			1	3.334	1.334	5.624	3.124	21.478
15		0.8	0	2.997	0.997	6.629	4.129	68.048
	5		1	3.108	1.108	6.353	3.853	56.722
			1	2.624	0.624	5.909	3.409	31.418
	12	1.5	0	3.574	1.574	5.486	2.986	684.217
	15		0	3.229	1.229	5.328	2.828	50.984
			0	2.514	0.514	5.107	2.607	26.497
25	9	0.8	0	2.798	0.798	6.247	3.747	74.818
			0	2.409	0.409	5.931	3.431	62.646
	18		1	2.418	0.418	5.858	3.358	39.903
		1.5	1	2.847	0.847	6.187	3.687	62.047
	25		0	2.541	0.541	5.842	3.342	55.872
			0	2.339	0.339	5.735	3.235	31.547
30	11	0.8	1	3.788	1.788	6.196	3.696	77.975
			0	2.252	0.252	5.893	3.393	72.652
	27		0	2.392	0.392	5.276	2.776	64.451
		1.5	1	3.647	1.647	5.984	3.484	63.598
	30		0	3.684	1.684	5.664	3.164	54.872
			0	2.314	0.314	5.107	2.607	33.714
50	23	0.8	0	4.033	2.033	6.004	3.504	78.195
			0	2.381	0.381	5.914	3.414	66.062
	33		1	1.621	0.379	5.165	2.665	21.834
		1.5	1	3.148	1.148	5.624	3.124	66.982
	50		1	2.841	0.841	5.742	3.242	62.175
			0	2.648	0.648	5.207	2.707	23.807

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 3, \quad \lambda = 1.2$)

n	r	T	H	$\hat{\alpha}$ MLE	$\hat{\alpha}$ Standard deviation	$\hat{\lambda}$ MLE	$\hat{\lambda}$ Standard deviation	$Co\,var(\hat{\alpha},\hat{\lambda})$
10	3	0.8	0	5.194	2.194	3.847	2.647	34.502
			0	5.043	2.043	3.627	2.427	21.454
	6		1	4.727	1.727	3.171	1.971	12.188
		1.5	1	5.324	2.324	3.954	2.754	36.257
			1	5.147	2.147	3.486	2.286	24.658
	10		0	4.254	1.254	3.092	1.892	18.547
15	7	0.8	0	5.057	2.057	3.908	2.708	65.414
			1	5.841	2.841	3.165	1.965	46.812
	11		0	5.286	2.286	3.025	1.825	33.817
		1.5	1	5.687	2.687	3.947	2.747	62.147
			1	5.471	2.471	3.547	2.347	45.147
	15		0	5.024	2.024	3.008	1.808	32.178
25	9	0.8	0	5.087	2.087	3.744	2.544	82.817
			0	4.854	1.854	3.566	2.366	71.282
	17		1	4.207	1.207	3.021	1.821	54.592
		1.5	1	4.987	1.987	3.648	2.448	76.214
			1	4.658	1.658	3.475	2.275	66.847
	25		0	4.198	1.198	3.147	1.947	51.278
30	19	0.8	1	5.420	2.420	3.856	2.656	88.008
			0	4.821	1.821	3.651	2.451	76.875
	27		0	4.182	1.182	2.971	1.771	44.718
		1.5	0	4.852	1.852	3.958	2.758	72.584
			0	4.719	1.719	3.742	2.542	54.572
	30		0	4.532	1.532	3.589	2.389	34.257
50	11	0.8	0	4.524	1.524	3.564	2.364	85.182
			0	4.421	1.421	3.084	1.884	73.984
	37		0	3.882	0.882	2.794	1.594	66.627
		1.5	1	4.872	1.872	3.667	2.467	71.248
			1	4.658	1.658	3.418	2.218	64.579
	50		1	3.652	0.652	3.014	1.814	54.214

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when $(\alpha = 3, \quad \lambda = 2)$

n	r	T	H	$\hat{\alpha}$ MLE	$\hat{\alpha}$ Standard deviation	$\hat{\lambda}$ MLE	$\hat{\lambda}$ Standard deviation	$Co\operatorname{var}(\hat{\alpha},\hat{\lambda})$
10	3	0.8	0	6.615	3.615	4.927	2.927	36.288
			1	6.721	3.721	4.588	2.588	20.454
	5		1	6.301	3.301	4.441	2.441	9.517
		1.5	0	6.874	3.874	5.147	3.147	38.147
			0	6.754	3.754	4.982	2.982	23.47
	10		0	6.247	3.247	4.247	2.247	11.478
15	6	0.8	1	5.952	2.952	5.159	3.159	69.043
			0	5.712	2.712	4.914	2.914	55.876
	11		0	5.819	2.819	4.249	2.249	38.874
		1.5	0	6.247	3.247	5.047	3.047	66.187
			0	5.984	2.984	4.871	2.871	58.498
	15		0	5.744	2.744	4.652	2.652	34.875
25	11	0.8	1	6.317	3.317	5.428	3.428	77.847
			1	6.074	3.074	5.001	3.001	58.571
	17		0	6.159	3.159	4.124	2.124	17.797
		1.5	0	5.984	2.984	5.147	3.147	68.497
			0	5.147	2.147	4.271	2.271	24.984
	25		0	5.078	2.078	4.014	2.014	14.872
30	18	0.8	1	5.618	2.618	5.081	3.081	69.187
			1	5.491	2.491	4.941	2.941	57.078
	23		0	5.107	2.107	4.028	2.028	23.174
		1.5	0	5.742	2.742	4.982	2.982	66.874
			1	5.324	2.324	4.751	2.751	52.987
	30		1	5.478	2.478	4.254	2.254	46.278
50	14	0.8	0	5.334	2.334	4.658	2.658	74.194
			0	5.238	2.238	4.748	2.748	60.827
	27		1	4.997	1.997	4.102	2.102	47.145
		1.5	0	5.347	2.347	4.627	2.627	70.842
			0	5.148	2.148	4.489	2.489	58.247
	50		1	5.004	2.004	4.309	2.309	41.872

Continued Table (1)

The maximum likelihood Estimator, the standard deviation and covariance of the Lomax with two parameter under hybrid censored sample when ($\alpha = 3, \quad \lambda = 2.5$)

n	r	T	H	$\hat{\alpha}$		$\hat{\lambda}$		$Co\,\text{var}$ $(\hat{\alpha},\hat{\lambda})$
				MLE	Standard deviation	MLE	Standard deviation	
	2	0.8	0	6.779	3.779	6.708	4.208	44.321
			0	6.628	3.628	6.574	4.074	30.911
10	8		1	6.409	3.409	5.814	3.314	20.427
		1.5	1	6.895	3.895	6.874	4.374	46.879
	10		1	6.664	3.664	6.547	4.047	38.584
			1	6.578	3.578	6.432	3.932	27.871
	9	0.8	0	6.689	3.689	6.841	4.341	58.814
			0	6.524	3.524	6.672	4.172	50.042
15	12		1	6.308	3.308	5.702	3.202	30.784
		1.5	1	6.874	3.874	6.654	4.154	66.589
	15		0	6.483	3.483	6.439	3.939	64.752
			0	6.147	3.147	6.307	3.807	41.058
	14	0.8	1	6.498	3.498	6.802	4.302	74.941
			0	6.384	3.384	6.471	3.971	62.603
25	17		0	5.287	2.287	5.629	3.129	39.913
		1.5	1	6.957	3.957	6.879	4.379	71.058
	25		1	6.821	3.821	6.847	4.347	58.195
			0	6.304	3.304	6.685	4.185	42.925
	22	0.8	1	5.692	2.692	6.794	4.294	62.054
			0	5.542	2.542	6.678	4.178	47.829
30	27		0	5.201	2.201	5.489	2.989	16.475
		1.5	0	6.741	3.741	6.524	4.024	78.284
	30		0	5.984	2.984	6.431	3.931	72.084
			1	5.721	2.721	6.278	3.778	61.894
	27	0.8	0	5.448	2.448	6.824	4.324	78.475
			0	5.707	2.707	6.671	4.171	66.942
50	35		1	5.334	2.334	5.391	2.891	11.313
		1.5	1	5.842	2.842	6.927	4.427	63.587
	50		1	5.328	2.328	6.14	3.64	52.108
			0	5.274	2.274	5.725	3.225	32.924

Table 2: The posterior mean and posterior variance of the Lomax with two parameter distribution under hybrid censored sample when ($\alpha = 1, \quad \lambda = 1.2$)

n	r	T	k	H	α		λ	
					Posterior mean	Posterior variance	Posterior mean	Posterior variance
10	3	0.8	0.5	0	0.495	0.927	1.017	0.719
				0	0.384	0.678	0.701	0.645
				1	0.258	0.457	0.608	0.201
	7		2	1	0.974	0.804	0.704	0.821
				0	0.712	0.714	0.651	0.643
				0	0.427	0.512	0.572	0.422
	10		3	0	1.151	0.685	0.671	0.765
				0	0.874	0.514	0.569	0.648
				0	0.622	0.338	0.408	0.491
15	6	0.8	0.5	0	0.234	1.087	1.01	0.604
				0	0.128	0.927	0.871	0.556
				0	0.087	0.814	0.691	0.493
	9		2	1	0.728	0.674	0.741	0.576
				1	0.669	0.582	0.687	0.477
				0	0.541	0.449	0.354	0.345
	15		3	1	0.742	0.742	0.948	0.785
				1	0.658	0.537	0.653	0.676
				0	0.432	0.338	0.527	0.527
25	14	0.8	0.5	1	0.958	0.881	0.885	0.692
				0	0.742	0.724	0.685	0.476
				0	0.689	0.628	0.359	0.215
	19		2	1	0.957	0.695	0.798	0.763
				0	0.759	0.584	0.619	0.792
				0	0.358	0.557	0.369	0.621
	25		3	0	0.482	0.725	0.437	0.911
				0	0.352	0.644	0.288	0.425
				0	0.241	0.559	0.119	0.307

H : terminate the experiment at min ($T, x_{(r)}$).

$$H = \begin{vmatrix} 0 & if & H = T \\ 1 & if & H = x_{(r)} \end{vmatrix}$$ and k : The prior

Continued Table (2)

The posterior mean and posterior variance of the Lomax with two parameter distribution under hybrid censored sample when ($\alpha = 1, \quad \lambda = 1.2$)

n	r	T	k	H	α		λ	
					Posterior mean	Posterior variance	Posterior mean	Posterior variance
30	13	0.8	0.5	0	0.992	0.761	0.718	0.901
				1	0.745	0.895	0.651	0.763
				1	0.604	0.612	0.581	0.522
	22		2	0	1.874	0.752	0.875	0.574
				0	1.685	0.594	0.669	0.947
				0	0.992	0.338	0.427	0.758
	30		3	1	0.852	0.628	0.609	0.827
				1	0.729	0.597	0.547	0.647
				0	0.635	0.448	0.489	0.558
50	26	0.8	0.5	0	0.875	0.762	0.671	0.758
				0	0.625	0.684	0.507	0.862
				0	0.558	0.449	0.407	0.647
	33		2	1	1.087	0.868	0.798	0.574
				0	0.995	0.698	0.664	0.482
				0	0.774	0.527	0.308	0.201
	50		3	1	0.957	0.728	0.827	0.728
				0	0.721	0.582	0.667	0.661
				0	0.408	0.448	0.524	0.337