A Study on the Mixture of Exponentiated-Weibull Distribution Part II (The Method of Bayesian Estimation)

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Abstract

The use of finite mixture distributions, to control for unobserved heterogeneity, has become increasingly popular among those estimating dynamic discrete choice models. One of the barriers to using mixture models is that parameters that could previously be estimated in stages must now be estimated jointly: using mixture distributions destroys any additive reparability of the log likelihood function. In this research, Bayesian estimators have been obtained for the parameters of the mixture of exponentiated Weibull distribution when sample is available from censoring scheme.

The maximum likelihood estimators of the parameters and the asymptotic variance covariance matrix have been obtained by Elshahat and Mahmoud (2016). Bayes and approximate Bayes (Lindley's form) estimators have been developed under squared error loss function as well as under LINEX loss function using non –informative type of priors for the parameters will be obtained. A numerical illustration for these new results is given.

Keywords and Phrases: Mixture of two exponentiated Weibull distribution (MTEW), Maximum likelihood estimation, Bayesian estimation, Approximate Bayesian estimation, Lindley approximation Monte-Carlo simulation.

1. Introduction

In probability and statistics, a mixture distribution is the probability distribution of a random variable whose values can be interpreted as being derived in a simple way from an underlying set of other random variables. In particular, the final outcome value is selected at random from among the underlying values, with a certain probability of selection being associated with each. Here the underlying random variables may be random vectors, each having the same dimension, in which case the mixture distribution is a multivariate distribution.

In cases where each of the underlying random variables is continuous, the outcome variable will also be continuous and its probability density function is sometimes referred to as a mixture density. The c.d.f. of a mixture is convex combination of the c.d.f's of its components. Similarly, the p.d.f. (pdf) of the mixture can also express as a convex combination of the p.d.f's of its components. The number of components in mixture distribution is often restricted to being finite, although in some cases the components may be countable. More general cases (i.e., an uncountable set of component distributions), as well as the countable case, are treated under the title of compound distributions.

A mixture is a weighted average of probability distribution with positive weights that add up to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution. Because of these weights, a mixture is in particular again a probability distribution. Probability distributions of this type arise when observed phenomena can be the consequence of two or more related, but usually unobserved phenomena, each of which leads to a different probability distribution. Mixtures and related structures often arise in the construction of probabilistic models. Pearson (1894) was the first researcher in the field of mixture distributions who considered the mixture of two normal distributions. After the study of Pearson (1894) there was long gap in the field of mixture distributions. Decay (1964) has improved the results of Pearson (1894), Hasselbled (1968) studied in greater detail about the finite mixture of distributions.

Life testing is an important method for evaluating component's reliability by assuming a suitable lifetime distribution. Once the test is carried out by subjecting a sample of items of interest to stresses and environmental conditions that typify the intended operating conditions, the lifetimes of the failed items are recorded. Due to time and cost constraints, often the test is stopped at a predetermined time (Type I censoring) or at a predetermined number of failures (Type II censoring). If each item in the tested sample has the same chance of being selected, then the equal probability sampling scheme is appropriate, and this has lead theoretically to the use of standard distributions to fit the obtained data. If the proper sampling frame is absent and items are sampled according to certain measurements such as their length, size, age or any other characteristic (for example, observing in a given sample of lifetimes that large values are more likely to be observed than small ones). In such a case the standard distributions cannot be used due to the presence of certain bias (toward large value in our example), and must be corrected using weighted distributions.

In life testing reliability and quality control problems, mixed failure populations are sometimes encountered. Mixture distributions comprise a finite or infinite number of components, possibly of different distributional types, that can describe different features of data. Some of the most important references that discussed different types of mixtures of distributions are Jaheen (2005b) and AL-Hussaini and Hussien (2011).

Finite mixture models have been used for more years, but have seen a real boost in popularity over the last decade due to the tremendous increase in available computing power. The areas of application of mixture models range from biology and medicine to physics, economics and marketing. On the one hand these models can be applied to data where observations originate from various groups and the group affiliations are not known, and on the other hand to provide approximations for multi-modal distributions [see Everitt and Hand (1981); Titterington et al. (1985); Maclachlan and Peel (2000), Shawky and Bakoban (2009) and Hanna and Abu-Zinadah (2010)].

We shall consider the exponentiated Weibull model, which includes as special case the Weibull and exponential models. The Exponentiated Weibull family EW [introduced by Mudholkar and Srivastava (1993) as extention of the Weibull family] contains distributions with bathtub shaped and unimodal failure rates besides a broader class of monoton failure rates. Applications of the exponentiated models have been carried out by some authors as Bain (1974); Gore et al. (1986); and Mudholkar and Hutson (1996).

Some statistical properties of this distribution (EW) are discussed by Singh et al. (2002) obtained Bayes estimators for the distribution parameters, reliability function and hazard function with type II censored sample under squared error loss function as well as under LINEX loss function. Nassar and Eissa (2004) obtained Bayes estimators of the two parameters EW distribution, reliability and failure rate functions using Bayes approximation form due to Lindley (1980) under the squared error loss and LINEX loss functions. Elshahat (2006), derived Bayes estimators for the two unknown shape parameters of the EW based on progressive type I interval censored sample. Salem and Abo-Kasem (2011) derived Bayes estimators for the two unknown shape parameters of the EW based on progressive hybrid censored sample. Approximate Bayes estimators for the two unknown shape parameters are drived by Elshahat (2008) based on Lindley (1980) and tierny and kadane (1986) and approximate credible intervals for the unknown parameters are obtained with progressive interval censoring. Ashour and afifiy (2008) derived maximum likelihood estimators of the parameters for EW with type II progressive interval censoring with random removals and their asymptotic variances. Elshahat and Mahmoud (2016) obtained maximum likelihood estimators of the parameters of the mixture of exponentiated Weibull distribution, reliability and hazard functions from type II censored samples.

Research outline

1. Obtain Bayesian and approximate Bayesian (Lindley's-approximation) estimators of the parameters with two different loss function, squared error loss function and LINEX loss function.
2. Monte Carlo simulation study will be done to compare between these estimators and the maximum likelihood, Bayesian and approximate Bayesian ones.
3. Applications of mixed models will be presented.
   In additional to above introduction, the research contains four sections. Section (2) is devoted to some important definitions and notation which will be used in the present research. In Section (3) the estimation of the Mixture of the exponentiated Weibull distribution parameters has been drived via Bayesian method. In Section (4) estimation of the Mixture of the exponentiated Weibull distribution parameters has been drived via approximate Bayesian method and in Section (5) Numerical illustration using real data and simulation technique has been used to study the behaviour of the estimators using the Mathcad (2011) packages.

2. Definitions and Notation

This section is devoted to some important definitions and notation which are used in the present research.

2.1. One Stage Single Censord Samples

In such experiment, one of the two main types of censoring schemes (type -I or type – II) is used. Suppose we put-n items on a test and terminate the experiment at pre-assigned time (*T*), the samples obtained from such an experiment are called "time-censored" samples. The number of failures r and all the failure times are random variables. If the experiment is terminated when pre-assigned fixed number of items, say $r<n$ have failed, the samples obtained from such an experiment are called "failure – censored" samples. The likelihood function of type-I (time censored) and type-II (failure censored) censored can be given as follow:

$$L(\underline{\theta}\,|\,\underline{t})=\frac{n!}{(n-r)!}\prod_{i=1}^{r} f(t_{(i)})[1-F(\Delta)]^{n-r}, \tag{1}$$

where $f(.)$ and $F(.)$ are the density and distribution functions, respectively.

When $\Delta=T$, then (1) reduces to the likelihood function of type-I censored, and when $\Delta=t_{(r)}$ then (1) reduces to the likelihood function of type–II censored. Type-I and type–II censoring corresponding to complete sampling when $n=r$.

2.2. Mixture Model

Mixtures of life distributions occur when two different causes of failure are present, each with the same parametric form of life distributions. In recent years, the finite mixtures of life distributions have proved to be of considerable interest both in terms of their methodological development and practical applications [see Titterington et al. (1985), Mclachlan and Basford (1988), Lindsay (1995), Mclachlan and Peel (2000) and Demidenko (2004)].

Mixture model is a model in which independent variables are fractions of a total. One of the types of mixture of the distribution functions which has its practical uses in a variety of disciplines.

Finite mixture distributions go back to end of the last century when Everitt and Hand (1981) published a paper on estimating the five parameters in a mixture of two normal distributions. Finite mixtures involve a finite number of components. It results from the fact that different causes of failure of a system could lead to different failure distributions, this means that the population under study is non-homogenous.

Suppose that T is a continuous random variable having a probability density function of the form:

$$f(t)=\sum_{j=1}^{k} p_j f_j(t), \qquad t>0, \quad k>1, \tag{2}$$

where $0 \le p_j \le 1, \quad j = 1,2,...,k$ and $\sum_{i=1}^{k} p_i = 1$. The corresponding c.d.f. is given by:

$$F(t) = \sum_{j=1}^{k} p_j F_j(t), \qquad t > 0, \quad k > 1, \tag{3}$$

where k is the number of components, the parameters $p_1, p_2, \dots, p_k$ are called mixing parameters, where p_i represent the probability that a given observation comes from population "i" with density $f_i(.)$, and $f_1(.), f_2(.), \dots, f_k(.)$ are the component densities of the mixture. When the number of components k=2, a two component mixture and can be written as:

$$f(t) = p \;\; f_1(t) + (1-p) f_2(t),$$

When the mixing proportion 'p' is closed to zero, the two component mixture is said to be not well separated.

Definition (1): Suppose that T and Y be two random variables. Let $F(t|y)$ be the distribution function of T given Y and G(y) be the distribution function of Y. The marginal distribution function $F(t)$, defined by:

$$F(t) = \int_{-\infty}^{\infty} F(t \mid y).dG(y), \tag{4}$$

is called a mixture of the distribution function $F(t|y)$ and $G(y)$ where $F(t|y)$ is known as the kernel of the integral and $G(y)$ as the mixing distribution .

A special case from definition (1) when the random variable Y is a discrete number of points $\{y_j, j = 1, 2, 3, \dots, k\}$ and G is discrete and assigns positive probabilities to only those values of Y; the integral (4) can be replaced by a sum to give a countable mixture:

$$F(t) = \sum_{j=1}^{\infty} g(y_j).F(t \mid y_j),$$

where $g(y_j)$ is the probability of y_j. If the random variable Y assumes only a finite number of distributions $\{y_j, j = 1, 2, 3, \dots, k\}$, Ahmed et al. (2013) have been used the finite mixture:

$$F(t) = \sum_{j=1}^{k} w_i F_j(t), \tag{5}$$

By differentiating (5) with respect to T, the finite mixture of probability density functions can be obtained as follows

$$f(t) = \sum_{j=1}^{k} w_i f_j(t), \tag{6}$$

where,

$$f_j(t) = \frac{dF_j(t)}{dt} = \frac{dF(t|y_j)}{dt} = f(t|y_j).$$

In (6), the masses w_j called the mixing proportions, they satisfy the conditions:

$$w_j \geq 0 \quad \text{and} \qquad \sum_{j=1}^{k} w_j = 1,$$

$F_j(.)$ *and* $f_j(.)$ are called the j^{th} component in the finite mixture of distributions (5) and probability density functions (6), respectively. Thus, the mixture of the distribution functions can be defined as a distribution function that is a linear combination of other distribution functions where all coefficients are non-negative and add up to 1.

The parameters in number of expressions (5) or (6) can be divided into three types. The first consists solely of k, the components of the finite mixture. The second consists of the mixing proportions w. The third consists of the component parameters (the parameters of F_j (.) or f_j(.)).

There is a number of papers dealing with 2-fold mixture models for times to failure modeling. For example, Jiang and Murthy (1995) characterized the 2-fold Weibull mixture models in terms of the Weibull probability plotting, and examined the graphical plotting approach to determine if a given data set can be modeled by such models. Ling and pan (1998) proposed the method to estimate the parameters for the sum of two three-parameter Weibull distributions. Based on these findings, a new procedure for the selection of population distribution and parameter estimation was presented.
The reliability of the mixture distributions is given by:

$$\begin{aligned} R(t) &= \sum_{j=1}^{k} p_j R_j(t), \qquad t>0, \;\; k>1 \\ &= p_1[1-F_1(t)] + p_2[1-F_2(t)], \end{aligned} \tag{7}$$

2.3. Exponentiated Weibull Distribution (EW)

Salem and Abo-Kasem (2011) derived EW distribution in the following details; the "exponentiated Weibull family" introduced by Mudholkar and Srivastava (1993) as extension of the Weibull family, contains distribution with bathtub shaped and unimodale failure rates besides a broader class of monotone failure rates. The applications of the exponentiated Weibull (EW) distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). Its properties have been studied in more detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003). The probability density function (p.d.f.), the cumulative distribution function (c.d.f.) and the reliability function of the exponentiated Weibull are given respectively by;

$$\mathrm{f(t)} = \alpha\theta(1-\mathrm{e}^{-\mathrm{t}^{\alpha}})^{\theta-1}\mathrm{e}^{-\mathrm{t}^{\alpha}}\mathrm{t}^{\alpha-1}, \qquad \mathrm{t}, \alpha, \theta > 0, \tag{8}$$

$$F(t,\alpha,\theta) = [1-e^{-t^{\alpha}}]^{\theta}, \tag{9}$$

and

$$R(t) = [1-(1-e^{-t^{\alpha}})^{\theta}], \qquad t>0. \tag{10}$$

Where α and θ are the shape parameters of the model (8). The distinguished feature of EW distribution from other life time distribution is that it accommodates nearly all types of failure rates both monotone and non-monotone (unimodal and bathtub). The EW distribution includes a number of distributions as particular cases: if the shape parameter θ = 1, then the p.d.f is that Weibull distribution, when α = 1 then the p.d.f is that Exponentiated Exponential distribution, if α = 1 and θ = 1 then the pdf is that Exponential distribution and if α = 2 then the p.d.f is that one parameters Burr-X distribution. Mudholkar and Hutson (1996) showed that the density of A random variable *T* is said to be followed a finite mixture distribution with k components, if the p.d.f, c.d.f and R(t) of *T* can be written as in the forms (2), (3) and (7) respectively [see Tittrerington et al. (1985)].

The hazard function (HF) of the mixture is given by;

$$H(t) = \frac{\sum_{j=1}^{k} p_j f_j(t)}{\sum_{j=1}^{k} p_j R_j(t)}, \tag{11}$$

function of the EW distribution is decreasing when αθ ≤ 1 and unimodal when αθ ≥ 1. The natural logarithm of the likelihood function (1) is given by;

$$l = \ln \quad L(\underline{\theta}) = \ln\left(\frac{n!}{(n-r)!}\right) + \sum_{i=1}^{r} \ln \ f(t_{i;n}) + (n-r)\ln R(t_{r:n}), \tag{12}$$

2.4. The Mixture of Two Exponentiated Weibull Distribution (MTEW)

In this chapter, we shall consider the mixture of two – component Exponentiated Weibull (MTEW) distribution. Some properties of the model with some graphs of the density and hazard functions are discussed. Elshahat and Mahmoud (2016) obtained the following maximum likelihood estimation under type II censored samples.

The failure of an item or a system can be caused by one or more than one cause of failure; it results that the density of time to failure can have one mode or multimodal shape and in that case, finite mixtures represent a good tool to model such phenomena. Suppose that two populations of the exponentiated Weibull (EW) distribution with two shapes parameters α and θ [see Mudholkar and Hutson (1996)] mixed in unknown proportions p and (1-p) respectively.

A random variable T is said to follow a finite mixture distribution with k components, if the p.d.f. of T can be written in the form (2) [see Titterington et al. (1985)]. Where $j = 1, \dots, k$, $f_j(t)$ the j^{th} p.d.f. component (8) and the mixing proportions, p_j, satisfy the conditions $0 < p_j < 1$ and $\sum_{j=1}^{k} p_j = 1$, the corresponding c.d.f., is given by (3), where $F_j(t)$ is the j^{th} c.d.f., component (9) , the reliability function (RF) of the mixture is given by (7), where $R_j(t)$ is the j^{th} reliability component (10) . The hazard function (HF) of the mixture is given by (11), where $f(t)$ and $R(t)$ are defined in (2) and (7) respectively.

Mixture of K EW components: Substituting (8) and (9) in (2) and (3), the p.d.f and c.d.f. of MTEW components are given respectively, by:

$$f(t)=\sum_{j=1}^{k} p_j\alpha_j\theta_j t^{\alpha-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{\theta_j-1}, \quad t>0,\ \ \alpha_j,\theta_j>0 \tag{13}$$

$$F(t)=\sum_{j=1}^{k} p_j(1-e^{-t^{\alpha_j}})^{\theta_j}, \quad t>0,\ \ \alpha_j,\theta_j>0 \tag{14}$$

where, for $j=1,...,k$, $0<p_j<1$ and $\sum_{j=1}^{k} p_j=1$.

By observing that R(t) = 1- F(t) and $\sum_{j=1}^{k} p_j=1$, the RF of MTEW distribution, $j=1,2,\dots,k$ components can be obtained from (7) and (10) as:

$$R(t)=\sum_{j=1}^{k} p_j[1-(1-e^{-t^{\alpha_j}})^{\theta_j}], \quad t>0,\ \ \alpha_j,\theta_j>0 \tag{15}$$

dividing (4.1.1) by (4.1.3), we obtain the HF of MTEW distribution as:

$$H(t)=\frac{\sum_{j=1}^{k} p_j\alpha_j\theta_j t^{\alpha-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{\theta_j-1}}{\sum_{j=1}^{k} p_j[1-(1-e^{-t^{\alpha_j}})^{\theta_j}]} \quad t>0,\ \ \alpha_j,\theta_j>0, \tag{16}$$

If k = 2, the p.d.f., c.d.f. RF and HF of MTEW distribution are then given, respectively by $f(t)=p\,\alpha_1\theta_1 t^{\alpha_1-1}e^{-t^{\alpha_1}}(1-e^{-t^{\alpha_1}})^{\theta_1-1}+\ (1-p)\alpha_2\theta_2 t^{\alpha_2-1}e^{-t^{\alpha_2}}(1-e^{-t^{\alpha_2}})^{\theta_2-1}$,

$F(t)=p(1-e^{-t^{\alpha_1}})^{\theta_1}+\ (1-p)\ (1-e^{-t^{\alpha_2}})^{\theta_2}$,

$R(t)=p(1-\left(1-e^{-t^{\alpha_1}}\right)^{\theta_1})+\ (1-p)\ (1-\left(1-e^{-t^{\alpha_2}}\right)^{\theta_2})$,

and, $H(t)=\dfrac{f(t)}{R(t)}$.

2.5. Maximum likelihood Estimation for the Unknown Parameters of MTEW under Type II Censored Sample

Suppose a type-II censored sample $\underset{\sim}{t}=(t_{1;n},t_{2;n},...,t_{r;n})$ where t_i the time of the i^{th} component to fail. This sample of failure times are obtained and recorded from a life test of n items whose life time have MTEW distribution, with p.d.f, and c.d.f given, respectively, by (13) and (14). The likelihood function in this case [see lawless (1982)] can be written as equation (2.13). Where $f(t_{i;n})$ and $R(t_{i;n})$ are given, respectively, by (13) and (15). The natural logarithm of the likelihood function (1) is given by equation (12). Assuming that the parameters, θ_1 and θ_2 are unknown, we differentiate the natural logarithm of the likelihood function (12) with respect to θ_j so the likelihood equations are given by

$$l_j = \frac{\partial l}{\partial \theta_j} = \sum_{i=1}^{r}\left[\frac{1}{f(t_{i:n})} \cdot \frac{\partial f(t_{i:n})}{\partial \theta_j}\right] + \frac{n-r}{R(t_{r:n})} \cdot \frac{\partial R(t_{r:n})}{\partial \theta_j}, \quad j = 1,2, \tag{17}$$

where l_j is the first derivatives of the natural logarithm of the likelihood function (12) with

respect to θ_j, from (13) and (15) respectively . we have: $\frac{\partial f(t_{i:n})}{\partial \theta_j}$

$$= p_j f_j(t_{i:n}) \mathrm{k}_j^*(t_{i:n}), \tag{18}$$

and

$$\frac{\partial R(t_{r:n})}{\partial \theta_j} = -p_j F_j(t_{r:n}) \omega^*(t_{r:n}), \tag{19}$$

where $p_1 = p$, $p_2 = 1 - p$,

$$k_j^*(t_{i:n}) = \theta_j^{-1} + \omega^*(t_{i:n}), \tag{20}$$

and

$$\omega^*(t_{i:n}) = \ln\left(1 - e^{-t_{i:n}{}^{\alpha_j}}\right), \tag{21}$$

Substituting (18) and (19) in (17), we obtain

$$l_j = p_j \left\{ \sum_{i=1}^{r} \zeta_j^{**}(t_{i:n}) k_j^*(t_{i:n}) - (n-r)\, \zeta_j^{***}(t_{r:n}) \omega^*(t_{r:n}) \right\} = 0, \tag{22}$$

where l_j is the first derivatives of the natural logarithm of the likelihood function (12) with respect to θ_j for $j = 1,2$, and $i = 1,2, ..., r$

$$\zeta_j^{**}(t_{i:n}) = \frac{f_j(t_{i:n})}{f(t_{i:n})}, \ \zeta_j^{***}(t_{r:n}) = \frac{F_j(t_{r:n})}{R(t_{r:n})}, \tag{23}$$

and $k_j^*(t_{i:n}), \omega^*(t_{i:n})$ are given by (20) and (21) respectively. Assuming that the parameters, α_1 and α_2 are unknown, we differentiate the natural logarithm of the likelihood function (12) with respect to α_j so the likelihood equations are given by:

$$\frac{\partial l}{\partial \alpha_j} = \sum_{i=1}^{r}\left[\left(\frac{1}{f(t_{i:n})} \ \frac{\partial f(t_{i:n})}{\partial \alpha_j}\right)\right] + \frac{n-r}{R(t_{r:n})} \ \frac{\partial R(t_{r:n})}{\partial \alpha_j}, \tag{24}$$

where $\dfrac{\partial l}{\partial \alpha_j}$ is the first derivatives of the natural logarithm of the likelihood function(12) with respect to α_j, from (13) and (15) respectively, we have

$$\frac{\partial f(t_{i:n})}{\partial \alpha_j} = p_j f_j(t_{i:n}) S_j(t_{i:n}), \tag{25}$$

and

$$\frac{\partial R(t_{r:n})}{\partial \alpha_j} = -p_j F_j(t_{r:n}) O_j(t_{r:n}), \tag{26}$$

where $p_1 = p$, $p_2 = 1 - p$,

$$S_j(t_{i:n}) = \alpha_j^{-1} + \ln(t_{i:n}) - t_{i:n}^{\alpha_j} \ln(t_{i:n}) + (\theta_j - 1)(1 - e^{-t^{\alpha_j}})^{-1} \times e^{-t_{(i:n)}^{\alpha_j}} t_{(i:n)}^{\alpha_j} \ln(t_{i:n}), \tag{27}$$

and

$$O_j(t_{r:n}) = \theta_j t_{(r:n)}^{\alpha_j} e^{-t_{(r:n)}^{\alpha_j}} (1 - e^{-t_{r:n}^{\alpha_j}})^{-1} \ln(t_{r:n}), \tag{28}$$

Substituting (25) and (26) in (24), we obtain

$$\frac{\partial l}{\partial \alpha_j} = p_j \left\{ \sum_{i=1}^{r} \zeta_j^{**}(t_{i:n}) S_j(t_{i:n}) - (n-r)\, \zeta_j^{***}(t_{r:n}) O_j(t_{r:n}) \right\} = 0, \tag{29}$$

where $\dfrac{\partial l}{\partial \alpha_j}$ is the first derivatives of the natural logarithm of the likelihood function (12) with respect to α_j for $j = 1,2$, and $i = 1,2,\ldots, r$,

$\zeta_j^{**}(t_{i:n})$, $\zeta_j^{***}(t_{r:n}), S_j(t_{i:n})$ and $O_j(t_{r:n})$ are given respectively by (23), (27) and (28).

The solution of the four nonlinear likelihood equations (22) and (29) yields the maximum likelihood estimate (MLE):

$\underline{\hat{\theta}} = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M}, \hat{\alpha}_{1,M}, \hat{\alpha}_{2,M})$ of $\underline{\theta} = (\theta_1, \theta_2, \alpha_1, \alpha_2)$,

The MLE's of R(t) and H(t) are given, respectively, by (15) and (16) after replacing $(\theta_1, \theta_2, \alpha_1, \alpha_2)$ by their corresponding MLE's, $\hat{\theta}_{1,M}, \hat{\theta}_{2,M}, \hat{\alpha}_{1,M}$ and $\hat{\alpha}_{2,M}$.

Since the equations (22) and (29) are clearly transcendental equations in $\hat{\theta}_j$ and $\hat{\alpha}_j$; that is, no closed form solutions are known they must be solved by iterative numerical techniques to provide solutions (estimators), $\hat{\theta}_j$ and $\hat{\alpha}_j$, in the desired degree of accuracy. To study the variation of the MLE's $\hat{\theta}_j$ and $\hat{\alpha}_j$, the asymptotic variance of these estimators are obtained.

The asymptotic variance covariance matrix of $\hat{\theta}_j$ and $\hat{\alpha}_j$ is obtained by inverting the information matrix with elements that are negative expected values of the second order derivatives of natural logarithm of the likelihood function, for sufficiently large samples, a reasonable approximation to the asymptotic variance covariance matrix of the estimators can be obtained as;

$$\left.\begin{bmatrix} -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_1^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_1\partial\theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_1\partial\alpha_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_1\partial\alpha_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_2\partial\theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_2^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_2\partial\alpha_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\theta_2\partial\alpha_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_1\partial\theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_1\partial\theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_1^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_1\partial\theta_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_2\partial\theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_2\partial\theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_2\partial\theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial\alpha_2^2} \end{bmatrix}^{-1}\right|_{\hat{\theta}_j,\hat{\alpha}_j}$$

$$\cong \begin{bmatrix} \mathrm{v}(\hat{\theta}_1) & cov(\hat{\theta}_1,\hat{\theta}_2) & cov(\hat{\theta}_1,\hat{\alpha}_1) & cov(\hat{\theta}_1,\hat{\alpha}_2) \\ cov(\hat{\theta}_1,\hat{\theta}_2) & \mathrm{v}(\hat{\theta}_2) & cov(\hat{\theta}_2,\hat{\alpha}_1) & cov(\hat{\theta}_2,\hat{\alpha}_2) \\ cov(\hat{\alpha}_1,\hat{\theta}_1) & cov(\hat{\alpha}_1,\hat{\theta}_2) & \mathrm{v}(\hat{\alpha}_1) & cov(\hat{\alpha}_1,\hat{\alpha}_2) \\ cov(\hat{\alpha}_2,\hat{\theta}_1) & cov(\hat{\alpha}_2,\hat{\theta}_2) & cov(\hat{\alpha}_2,\hat{\alpha}_1) & \mathrm{v}(\hat{\alpha}_2) \end{bmatrix} \tag{30}$$

The appropriate (30) is used to derive the 100 (1-γ) % confidence intervals of the parameters as in following forms :

$$\hat{\theta}_j \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\theta}_j)}\ ,\hat{\alpha}_j \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\alpha}_j)}\,, \qquad for\ j=1,2$$

where, $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ percentile of the standard normal distribution.

The asymptotic variance – covariance matrix will be obtained by inverting the information matrix with the elements that are negative of the observed values of the second order derivate of the logarithm of the likelihood functions .using the logarithm of the likelihood functions (12), the elements of the information matrix are given by:

$$l_{12} = l_{21} = \frac{\partial l_1}{\partial\theta_2} = \frac{\partial l_2}{\partial\theta_1}$$

$$= -pp_2\left\{\sum_{i=1}^{r}\varphi^*(t_{i:n}) + (n-r)\omega^{*2}(t_{r:n})\Psi^*(t_{r:n})\right\}, \tag{31}$$

where l_1 is the first derivatives of the natural logarithm of the likelihood function (12) with respect to $\theta_1$and l_2 is the first derivatives of the natural logarithm of the likelihood function (12) with respect to θ_2.

and for i = 1, 2, …, r

where

$$\varphi^*(t_{i:n}) = k_1^*(t_{i:n})k_2^*(t_{i:n})\,\zeta_1^{**}(t_{i:n})\ \zeta_2^{**}(t_{i:n}), \tag{32}$$

$$\Psi^*(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n}), \tag{33}$$

$$l_{jj} = \frac{\partial l_j}{\partial \theta_j} = -p_j \left\{ \sum_{i=1}^{r} A_j^* \, (t_{i:n}) + (n-r)\omega^*(t_{r:n}) B_j^*(t_{r:n}) \right\}, \tag{34}$$

$$A_j^*(t_{i:n}) = \zeta_{\ j}^{**}(t_{i:n})\theta_j^{-2} - \ p_s k_j^{*^2}(t_{i:n})\, \zeta_{\ j}^{**}(t_{i:n})\, \zeta_{\ s}^{**}(t_{i:n}),$$

and

$$B_j^*(t_{r:n}) = \zeta_{\ j}^{***}(t_{r:n})\omega^*(t_{r:n}) + p_j\, \zeta_{\ j}^{***^2}(t_{r:n})\omega^*(t_{r:n}),$$

Where l_{jj} is the second derivatives of the natural logarithm of the likelihood function (12) with respect to θ_j, for $j = 1, 2,\ s = 1, 2,\ j \neq s$ the functions $k_j^*(.)$ and ω^* (.) are given by (20) and (21), $\zeta_{\ j}^{**}(.)$ and $\zeta_{\ j}^{***}(.)$ by (23).

$$\frac{\partial^2 l}{\partial \alpha_1 \alpha_2} = \frac{\partial^2 l}{\partial \alpha_2 \alpha_1}$$
$$= -p_1 p_2 \{\textstyle\sum_{i=1}^{r} \varphi^{**}(t_{i:n}) + (n-r)\Psi^{**}(t_{r:n})\}\,, \tag{35}$$

where

$$\varphi^{**}(t_{i:n}) = \zeta_{\ 1}^{**}(t_{i:n})\, \zeta_{\ 2}^{**}(t_{i:n}) S_1(t_{i:n}) S_2(t_{i:n}), \tag{36}$$

$$\Psi^{**}(t_{r:n}) = \zeta_{\ 1}^{***}(t_{r:n})\, \zeta_{\ 2}^{***}(t_{r:n}) O_1(t_{r:n}) O_2(t_{r:n}), \tag{37}$$

$$\frac{\partial^2 l}{\partial \alpha_j^2} = -p_j \left\{ \textstyle\sum_{i=1}^{r} A_j^{**}(t_{i:n}) + (n-r) B_j^{**}(t_{r:n}) \right\}, \tag{38}$$

where

$$A_j^{**}(t_{i:n}) = \zeta_{\ j}^{**}(t_{i:n}) S_j^{\setminus}(t_{i:n}) - \ p_s S_j^2(t_{i:n})\, \zeta_{\ j}^{**}(t_{i:n})\, \zeta_{\ s}^{**}(t_{i:n})\,,\ \ j=1,2,\ s=1,2,\ s\neq j,$$

$$B_j^{**}(t_{r:n}) = \ \zeta_{\ j}^{***}(t_{r:n})\ \ O_j(t_{r:n})\, Z_j(t_{r:n}) + \tau^{\ **}(t_{r:n})\ \ O_j(t_{r:n})$$

$$Z_j(t_{r:n}) = \ln(t_{r:n}) - t_{r:n}^{\alpha_j} \ln(t_{r:n}) - t_{r:n}^{\alpha_j} e^{-t_{r:n}^{\alpha_j}} \ln(t_{r:n})\, (1 - e^{-t_{r:n}^{\alpha_j}})^{-1}\,,$$

$$S_j^{\setminus}(t_{i:n}) = -\alpha_j^{-2} + t_{i:n}^{\alpha_j} \ln^2(t_{i:n}) - \{ (\theta_j - 1) \ln(t_{i:n})\, [e^{-2t^{\alpha_j}} t^{2\alpha_j} \ln(t_{i:n})(1 - e^{-t_{i:n}^{\alpha_j}})^{-2}]\},$$

and

$$\frac{\partial^2 l}{\partial \theta_1 \partial \alpha_1} = p_1 \left\{ \sum_{i=1}^{r} A_1^{\Delta}(t_{i;n}) + (n-r) B_1^{\Delta}(t_{r;n}) \right\}, \tag{39}$$

where

$$A_1^{\Delta}(t_{i:n}) = p_2\, \zeta_{\ 1}^{**}(t_{i:n})\, \zeta_{\ 2}^{**}(t_{i:n}) k_1^*(t_{i:n}) S_1(t_{i:n}) + \ \zeta_{\ 1}^{**}(t_{i:n}) D_1^*(t_{i:n}),$$

$$B_1^{\Delta}(t_{r:n}) = \ \zeta_{\ 1}^{***}(t_{r:n}) D_1^*(t_{r:n}) + \tau_1^{\Delta}(t_{r:n})\omega^*(t_{r:n}),$$

where

$$D_1^*(t_{i:n}) = e^{-t_{i:n}^{\alpha_j}} t_{i:n}^{\alpha_1} (1 - e^{-t_{i:n}^{\alpha_1}})^{-1} \ln(t_{i:n}),$$

$$\tau_1^{\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})O_1(t_{r:n}) + p_1 \zeta_1^{***^2}(t_{r:n})O_1(t_{r:n}), \tag{40}$$

$$\frac{\partial^2 l}{\partial\theta_2 \partial\alpha_1} = -p_1 p_2 \left\{\sum_{i=1}^{r} A_1^{\Delta\Delta}(t_{i:n}) + (n-r)\, B_1^{\Delta\Delta}(t_{r:n})\right\}, \tag{41}$$

where

$$A_1^{\Delta\Delta}(t_{i:n}) = -\zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})k_2^*(t_{i:n})S_1(t_{i:n}),$$

$$B_1^{\Delta\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n})O_1(t_{r:n})\omega^*(t_{r:n}),$$

$$\frac{\partial^2 l}{\partial\theta_1 \partial\alpha_2} = -p_1 p_2 \left\{\sum_{i=1}^{r} A_2^{\Delta\Delta}(t_{i:n}) + (n-r)\, B_2^{\Delta\Delta}(t_{r:n})\right\}, \tag{42}$$

where

$$A_2^{\Delta\Delta}(t_{i:n}) = \zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})K_1^*(t_{i:n})S_2(t_{i:n}),$$

$$B_2^{\Delta\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n})O_2(t_{r:n})\omega^*(t_{r:n}),$$

$$\frac{\partial^2 l}{\partial\theta_2 \partial\alpha_2} = P_2 \left\{\sum_{i=1}^{r} A_2^{\Delta}(t_{i:n}) + (n-r)\, B_2^{\Delta}(t_{r:n})\right\}, \tag{43}$$

$$A_2^{\Delta}(t_{i:n}) = P_1 \zeta_1^{**}(t_{i:n})\zeta_2^{**}(t_{i:n})K_2^*(t_{i:n})S_2(t_{i:n}) + \zeta_2^{**}(t_{i:n})D_2^*(t_{i:n}),$$

$$B_2^{\Delta}(t_{r:n}) = \zeta_2^{***}(t_{r:n})D_2^*(t_{i:n}) + \tau_2^{\Delta}(t_{r:n})\omega^*(t_{r:n}),$$

$$\tau_2^{\Delta}(t_{r:n}) = \zeta_2^{***}(t_{r:n})O_2(t_{r:n}) + p_2 \zeta_2^{***^2}(t_{r:n})O_2(t_{r:n}),$$

and

$$D_2^*(t_{i:n}) = e^{-t_{i:n}^{\alpha_2}} t_{i:n}^{\alpha_2} \ln(t_{i:n})(1 - e^{-t_{i:n}^{\alpha_2}})^{-1},$$

For *j=1,2* the functions S_j (.) and O_j(.) are as given by (27) and(28), ζ_j^{**}(.) and ζ_j^{***}(.) by(23).

2.6. LINEX Loss Function

Linear- Exponential loss function (LINEX) was proposed by Varian (1975) in the context of real-estate valuations. Klebanov (1976a) derived this loss function in developing his theory of loos function satisfying a Rao-Blackwell condition. The name LINEX is justified by the fact that this loss function rises approximately linearly on one side of zero and approximately exponentially on the other side, Zellner (1986) provided a detailed study of LINEX loss function and initiated a good deal of interest in estimation under this loss function. The LINEX loss function may be expressed as:

$$\mathrm{L}(\Delta) \propto e^{c\Delta} - c\Delta - 1, \quad c \neq 0, \tag{45}$$

where $\Delta = \tilde{\theta} - \theta$. the sign and magnitude of the shape parameter c reflects the direction and degree of asymmetry respectively. (If c > 0, the overestimation is mor serious than underestimation, and vice – versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function is:

$$E_{\theta}[L(\tilde{\theta}-\theta)] \propto \exp(c\tilde{\theta})$$
$$=[\exp-(c\theta)]-c(\tilde{\theta}-E_{\theta}(\theta))-1, \tag{46}$$

where $E_{\theta}(.)$ denoting posterior expectation with respect to the posterior density of θ. By a result of Zellner (1986), the (unique) Bayes estimator of θ, denoted by $\tilde{\theta}_{\mathrm{L}}$ under the LINEX loss is the value $\tilde{\theta}$ which minimizes (2.8) is given by

$$\tilde{\theta}_{\mathrm{L}} = -\frac{1}{c}\ln\{E_{\theta}[\exp(-c\theta)]\} \ , \tag{47}$$

Provided that the expectation $E_{\theta}[\exp(-c\theta)$, exists and is finite [see Shawky and Bakoban (2009)].

2.7. Approximate Bayesian Methods

When the posterior distribution takes a ratio form that involves integration in the denominator and cannot be reduced to a closed form, the evaluation of the posterior expectation for obtaining the Bayes estimators will be tedious. Among the various methods suggested to approximate the ratio of integrals in this case, perhaps the simplest one is Lindley's (1980) approximation method

Lindley's procedure was developed by Lindley (1980) to evaluate the posterior expectation of $\phi(\underline{\theta})$ as

$$\tilde{\phi} = E[\phi(\underline{\theta}) \,|\, T=t] = \int_{\Omega} \phi(\underline{\theta})\pi(\underline{\theta}\,|\,\underline{t})d\theta$$
$$= \frac{\int_{\Omega} \phi(\underline{\theta})L(\underline{\theta}\,|\,\underline{t})\pi(\underline{\theta})d\underline{\theta}}{\int_{\Omega} L(\underline{\theta}\,|\,\underline{t})\pi(\underline{\theta})d\underline{\theta}}, \tag{48}$$

where $\underline{\theta} = (\theta_1, \theta_2, ..., \theta_n) \in \Omega$ is a vector of parameters, $L(\underline{\theta}\,|\,\underline{t})$ is the likelihood function and $\pi(\theta)$ is the prior density of θ.

This procedure has been used by many authors to obtain Bayes estimators of the parameters of some distributions. See for example Soliman (2000), Jaheen (2005b).

The ratio of the integrals (48) may thus be approximated by using a form due to Lindley (1980) which reduces, in the case of two parameters, to the form:

$$\tilde{\phi} = \phi(\underline{\theta}) + \frac{S}{2} + \rho_1 S_{12} + \rho_2 S_{21} + \frac{1}{2}\left[\overset{*}{\underset{30}{l}} v_{12} + \overset{*}{\underset{21}{l}} c_{12} + \overset{*}{\underset{12}{l}} c_{21} + \overset{*}{\underset{03}{l}} v_{21}\right], \tag{49}$$

where $\underline{\theta} = (\theta_1, \theta_2), S = \sum_{i=1}^{2} \sum_{i=1}^{2} \phi_{ij}\sigma_{ij}, i, j = 1,2,\ \phi_{ij} = \dfrac{\partial^2 \phi}{\partial\theta_i\, \partial\theta_j}, \sigma_{\mathrm{ij}} = (i, j)$ th element in the matrix Σ, where $\Sigma = -[J(\theta)]^{-1}$, $J(\underline{\theta}) = [l_{ij}], l_{ij} = \dfrac{\partial^2 l}{\partial\theta_i \partial\theta_j}, l = \ln L(\underline{\theta} \,|\, \underline{t}),$ is the natural logarithm of the likelihood function.

For $i \neq j, S_{ij} = \phi_i\sigma_{ii} + \phi_j\sigma_{ji}, v_{ij} = (\phi_i\sigma_{ii} + \phi_j\sigma_{ij})\sigma_{ii}, c_{ij} = 3\phi_i\sigma_{ii}\sigma_{ij} + \phi_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$,

where $\phi_i = \dfrac{\partial\ \phi}{\partial\theta_i}$ and $\rho_i = \dfrac{\partial\rho}{\partial\theta_i}, \rho = \ln[\pi(\underline{\theta})]$. All functions in equation (29) are evaluated at $(\hat{\theta}_1, \hat{\theta}_2)$, the mode of the posterior density [see Salem and Nassar (2011)].

3. Bayesian Estimators for the Unknown parameters of MTEW under Type II Censored Sample

In this section, Bayesian method is used to obtain the estimators and posterior variance of the unknown parameters of finite mixture of two Exponentiated Weibull (MTEW) distribution. Bayesian risks are also obtained using the symmetric squared error loss. Moreover, for illustration, numerical examples are given.

3.1. Estimation under squared error loss function: (when α_1 and α_2 are known)

The likelihood function of the total life times $t_{1;n}$, $t_{2;n}$, …, $t_{r;n}$, where $t_{i;n}$ is the i^{th} component to fail. Considering the type II censored case this sample of failure times are obtained and recorded from a life test of n items independent and identically distributed, the Bayes estimators for θ_j and α_j using the likelihood function given by (1).

Assumed that the parameters have independent prior distribution and let the non informative prior (NIP) for θ_1 and θ_2 are respectively given by.

$\pi(\theta_1) \propto \theta_1^{-1}$, $\theta_1 > 0$ and $\pi(\theta_2) \propto \theta_2^{-1}$, $\theta_2 > 0$

Consequently, the joint (NIP) will be as follows:

$$\pi(\underline{\theta}) = (\theta_1, \theta_2)^{-1}, \quad \theta_1, \theta_2 > 0 \tag{50}$$

Multiplying (1) by (50), the joint posterior density of θ_1 and θ_2 given the data will be

$$g(\theta_1, \theta_2 \,/\, \underline{t}) = Q_2 \,/\, Q_1, \qquad \theta_1, \theta_2 > 0, \tag{51}$$

where, Q_1 is normalized constant equal to

$$Q_1 = \int_0^\infty \int_0^\infty L\left(\underline{\theta}|\underline{t}\right).\pi(\theta_1, \theta_2)\, d\theta_1 d\theta_2, \tag{52}$$

$$Q_2 = (\theta_1, \theta_2)^{-1} L\left(\underline{\theta}|\underline{t}\right), \tag{53}$$

Now, the marginal posterior of θ_1 can be obtained as.

$$g_3\left(\theta_1 \middle| \theta_2, \underline{t}\right) = Q_6 \,/\, Q_1, \quad \theta_1 > 0\,, \tag{54}$$

$$Q_6 = (\theta_1^{-1}) \int_0^\infty (\theta_2)^{-1} \mathrm{L}(\underline{\theta}|t) \mathrm{d}\theta_2,$$

and finally, the marginal posterior of θ_2 can be obtained as.

$$g_4(\theta_2|\theta_1, t) = Q_7|Q_1, \quad \theta_2 > 0\,, \tag{55}$$

where

$$Q_7 = (\theta_2)^{-1}\int_0^\infty (\theta_1)^{-1}\mathrm{L}(\underline{\theta}|t)d\theta_1, \tag{56}$$

It is well known that under a squared error loss function, the Bayes estimator of a parameter will be its posterior expectation. To obtain the posterior mean and posterior variance of the unknown parameters, non – tractable integrals will be confronted. So, in this problem numerical integration is required. Then, both the posterior mean and posterior variance of the unknown parameters (θ_1, θ_2) are expressed as follow.

$$\mathrm{E}\left(\theta_1\middle|\theta_2, \underline{t}\right) = \tilde{\theta}_{1,s} = Q_{12}/Q_1, \tag{57}$$

$$\mathrm{Var}\left(\tilde{\theta}_{1,s}\middle|\theta_2, \underline{t}\right) = Q_{13}/Q_1\,, \tag{58}$$

$$\mathrm{E}\left(\theta_2\middle|\theta_1, \underline{t}\right) = \tilde{\theta}_{2,s} = Q_{14}/Q_1, \tag{59}$$

and

$$\mathrm{Var}\left(\tilde{\theta}_{2,s}\middle|\theta_1, \underline{t}\right) = Q_{15}/Q_1, \tag{60}$$

where

$$Q_{12} = \int_0^\infty\int_0^\infty \theta_1 . Q_2\ d\theta_1\, d\theta_2, \tag{61}$$

$$Q_{13} = \int_0^\infty\int_0^\infty (\tilde{\theta}_{1,s} - \theta_1)^2\,, Q_2\, d\theta_1 d\theta_2, \tag{62}$$

$$Q_{14} = \int_0^\infty\int_0^\infty \theta_2 . Q_2\ d\theta_2 d\theta_1\,, \tag{63}$$

and

$$Q_{15} = \int_0^\infty\int_0^\infty (\tilde{\theta}_{2,s} - \theta_2)^2 . Q_2\, d\theta_2 d\theta_1\,, \tag{64}$$

Equations (57) to (60) are very complicated for solving. An iterative procedure is applied to solve these equations numerically using mathcad (2011).

3.2. Estimation under LINEX Loss Function: (when α_1 and α_2 are known)

The Bayes estimators for the parameters θ_1, θ_2 of MTEW distribution can be expressed as:

$$\tilde{\theta}_{1,L} = -\frac{1}{cc}\ln E\left(e^{-cc\theta_1}\right), \qquad cc \neq \theta,$$

$$= -\frac{1}{cc}\ln[\int_0^\infty e^{-cc\theta_1} g_3\,(\theta_1|\theta_2, \underline{t})d\theta_1]$$

$$= -\frac{1}{cc}\ln\left[\int_0^\infty e^{-cc\theta_1}\frac{\theta_1\int_0^\infty {\theta_2}^{-1} L\left(\underline{\theta}|t\right)d\theta_2}{Q_1}\, d\theta_1\right],$$

$$\tilde{\theta}_{2,L} = -\frac{1}{cc}\, ln\, E\left(e^{-cc\theta_2}\right), \qquad cc\ \neq \theta,$$

$$= -\frac{1}{cc}\ln[\int_0^\infty e^{-cc\theta_2} g_4(\theta_2|\theta_1, \underline{t})\, d\theta_2]\,,$$

$$= -\frac{1}{cc}\ln[\int_0^{\infty} e^{-cc\theta_2}\frac{\theta_2\int_0^{\infty}\theta_1{}^{-1}L(\underline{\theta}|t)d\theta_1}{Q_1}d\theta_2\,],$$

4. Approximate Lindley Bayesian Method

4.1. Lindley's Procedure: (when α_1 and $\alpha_2$are known).

Applying Lindley's form (49), we first obtain the elements σ_{ij}, which can be obtained as $\sigma_{11} = \frac{-l_{22}}{D}, \sigma_{22} = \frac{-l_{11}}{D}$ and

$$\sigma_{12} = \sigma_{21} = \frac{l_{12}}{D}, \tag{65}$$

Where

$$D = l_{11}l_{22} - l_{12}{}^2, \tag{66}$$

$l_{12} = l_{21}$ given in (31), and $l_{jj} = \frac{\partial^2 l_j}{\partial\theta_j^2}$ given in (34), ρ = ln $\pi(\underline{\theta})$, $\pi(\underline{\theta})$ is given by (50).

and

$l_{30}^* = \frac{\partial l_{11}}{\partial\theta_1}, l_{21}^* = \frac{\partial l_{21}}{\partial\theta_1}, l_{12}^* = \frac{\partial l_{21}}{\partial\theta_2}$ and $l_{03}^* = \frac{\partial l_{22}}{\partial\theta_2}$

Further more

$$l_{30}^* = \frac{\partial l_{11}}{\partial\theta_1} = -p_1\left\{\sum_{i=1}^r \frac{\partial A_1{}^*(t_{i:n})}{\partial\theta_1} + (n-r)\omega^*(t_{r:n})\frac{\partial B_1{}^*(t_{r:n})}{\partial\theta_1}\right\}, \tag{67}$$

$$l_{03}^* = \frac{\partial l_{22}}{\partial\theta_2} = -p_2\left\{\sum_{i=1}^r \frac{\partial A_2{}^*(t_{i:n})}{\partial\theta_2} + (n-r)\omega^*(t_{r:n})\frac{\partial B_2{}^*(t_{r:n})}{\partial\theta_2}\right\}, \tag{68}$$

$$l_{21}^* = \frac{\partial l_{21}}{\partial\theta_1} = -p_1p_2\left\{\sum_{i=1}^r \frac{\partial \varphi^*(t_{i:n})}{\partial\theta_1} + (n-r)\omega^{*2}(t_{r:n})\frac{\partial \Psi^*(t_{r:n})}{\partial\theta_1}\right\}, \tag{69}$$

$$l_{12}^* = \frac{\partial l_{21}}{\partial\theta_2} = -p_1p_2\left\{\sum_{i=1}^r \frac{\partial \varphi^*(t_{i:n})}{\partial\theta_2} + (n-r)\omega^{*2}(t_{r:n})\frac{\partial \Psi^*(t_{r:n})}{\partial\theta_2}\right\}, \tag{70}$$

where, for j, s= 1, 2 and j $\neq$s,

$$\frac{\partial\, A^*{}_{j(t_{i=n})}}{\partial\theta_j} = \theta_j^{-2}\,\zeta_j^{**`}\,(t_{i:n}) - 2\theta^{-3}\,\zeta_j^{**}\,(t_{i:n})$$
$$- p_s\,\zeta_s^{**}(t_{i:n})\left\{k_j^{*^2}(t_{i:n})\,\zeta_j^{**`}\,(t_{i:n}) + 2k_j^*(t_{i:n})k_j^{*`}(t_{i:n})\,\zeta_j^{**}\,(t_{i:n})\right\},$$

$$\frac{\partial B^*{}_j(t_{i=n})}{\partial\theta_j} = \left\{\omega^*(t_{r:n})\,\zeta_j^{***`}\,(t_{r:n}) + 2p_j\,\zeta_j^{***`}\,(t_{r:n})\,\zeta_j^{***}\,(t_{r:n})\omega^*(t_{r:n})\right\},$$

$$\frac{\partial\varphi^*(t_{i:n})}{\partial\theta_j} = k_s^*(t_{i:n})\,\zeta_s^{**}(t_{i:n})\left\{k_j^*(t_{i:n})\,\zeta_j^{**`}\,(t_{i:n}) + \zeta_j^{**}\,(t_{i:n})k_j^{*`}(t_{i:n})\right\}, \tag{71}$$

$$\frac{\partial\Psi^*(t_{r:n})}{\partial\theta_j} = \zeta_s^{***}\,(t_{r:n})\,\zeta_j^{***`}\,(t_{r:n}) \quad , \tag{72}$$

For j, s = 1,2 and j $\neq$ s

$$\zeta_j^{**`}\,(t_{i:n}) = k_j^*(t_{i:n})\,\zeta_j^{**}\,(t_{i:n})P_s\,\zeta_s^{**}\,(t_{i:n}), \tag{73}$$

$$\zeta_j^{***`}(t_{r:n}) = p_j \zeta_j^{***^2}(t_{r:n})\omega^*(t_{r:n}) + \zeta_j^{***}(t_{r:n})\omega^*(t_{r:n}), \tag{74}$$

and

$$k^{*'}(t_{i:n}) = -\frac{1}{\theta_j^2}, \tag{75}$$

4.2. Approximate Bayes (Lindley's Procedure) Estimation Under Squared Error Loss Function: (when α_1 and α_2 are known)

Now, we shall study the case when the parameters α_1 and α_2 are known, Suppose that the mixing proportion, $p_j, j = 1, 2$ and $\alpha_j, j = 1, 2$ are known.

Approximate Bayes Estimation of the Vector of Parameters

The two parameters, θ_1 and θ_2 can be approximately estimated using Lindley's approximation from (49) as follow:

i. The Bayes estimator of the parameter (θ_1)

Set $\emptyset(\underline{\theta}) = \theta_1$ in (49) , then

$\emptyset_1 = \dfrac{\partial\emptyset}{\partial\theta_1} = 1 , \emptyset_2 = \dfrac{\partial\emptyset}{\partial\theta_2} = 0$,

for $i, j = 1,2$, $\quad \emptyset_{ij} = \frac{\partial^2\emptyset}{\partial\theta_i\partial\theta_j} = 0$,

S = 0, $S_{12} = \sigma_{11}, \ S_{21} = \sigma_{12}$,

$v_{12} = \sigma_{11}^2 , \ v_{21} = \sigma_{21}\sigma_{22}$,

$c_{12} = 3\sigma_{11}\sigma_{12}$ and $c_{21} = \sigma_{22}\sigma_{11} + 2\sigma_{21}^2$

Substituting the above functions and [67 – 70] in (49) yields, the

Bayes estimator under squared error loss function, $\tilde{\tilde{\theta}}_{1,s}$ of θ_1

ii. The Bayes Estimator of the Parameter θ_2:

Set $\emptyset(\underline{\theta}) = \theta_2$ in (49) then

$\emptyset_1 = \dfrac{\partial\emptyset}{\partial\theta_1} = 0 , \emptyset_2 = \dfrac{\partial\emptyset}{\partial\theta_2} = 1$,

for i, j = 1, 2 , $\quad \emptyset_{ij} = \frac{\partial^2\emptyset}{\partial\theta_i\partial\theta_j} = 0$

S = 0 , $S_{12} = \sigma_{21}, \ S_{21} = \sigma_{22}, v_{12} = \sigma_{12}\sigma_{11} , \ v_{21} = \sigma_{22}^2$,

$c_{12} = \sigma_{11}\sigma_{22} + 2\sigma_{12}^2$ and $c_{21} = 3\sigma_{22}\sigma_{21}$

Substituting the above functions and [67 – 70] in (49) yields, the

Bayes estimator under squared error loss function , $\tilde{\tilde{\theta}}_{2,s}$ of θ_1

The Bayes estimator of RF:

Set $\emptyset(\underline{\theta})$ = R(t) in (49)

where R(t) is given as in (4.1.3) , then , for j=1,2

$$\emptyset_j = \frac{\partial\emptyset}{\partial\theta_j} = \frac{\partial \mathrm{R(t)}}{\partial\theta_j} = -P_j F_j(t)\omega^*(t),\ \emptyset_{12} = \emptyset_{21} = \frac{\partial^2\emptyset}{\partial\theta_1\,\partial\theta_2} = \frac{\partial^2 R(t)}{\partial\theta_1\,\partial\theta_2} = 0, \qquad (76)$$

and

$$\emptyset_{jj} = \frac{\partial^2\emptyset}{\partial\theta_j^2} = \frac{\partial^2 R(t)}{\partial\theta_j^2} = -P_j f_j(t)\omega^*(t), \qquad (77)$$

where $f_j(t)$, $F_j(t)$ and $\omega^*(t)$ are as given, respectively, by (8), (9) and (21) substituting (76), (77) and (67 - 70) in (49) yields the Bayes estimator under squared error loss function, $\tilde{\tilde{R}}_s$, of R(t).

The Bayes estimator of HF:

Set $\emptyset(\underline{\theta})$ = H(t) in (2.12), where H(t) is given as in (16) , then , for j=1,2 ,

$$\emptyset_j = \frac{\partial\emptyset}{\partial\theta_j} = \frac{\partial \mathrm{H(t)}}{\partial\theta_j} = \frac{1}{(R(t))^2}\left\{R(t)\frac{\partial f(t)}{\partial\theta_j} - f(t)\frac{\partial R(t)}{\partial\theta_j}\right\}$$

$$= \frac{P_j}{(R(t))^2}\left\{R(t)f_j(t)k_j^*(t) + f(t)F_j(t)\omega^*(t)\right\}, \qquad (78)$$

$$\emptyset_{jj} = \frac{\partial^2\emptyset}{\partial\theta_j^2} = \frac{\partial\emptyset_j}{\partial\theta_j} = \frac{E_1 - E_2}{(R(t))^4}, \qquad (79)$$

$$E_1 = (R(t))^2\left\{\frac{\partial R(t)}{\partial\theta_j}.\frac{\partial f(t)}{\partial\theta_j} + R(t)\frac{\partial^2 f(t)}{\partial\theta_j^2} - \left[\frac{\partial f(t)}{\partial\theta_j}.\frac{\partial R(t)}{\partial\theta_j} + f(t)\frac{\partial^2 R(t)}{\partial\theta_j^2}\right]\right\}$$

$$= p_j(R(t))^2\left\{R(t)f_j(t)\left[k_j^{*`}(t) + k_j^{*^2}(t) + \omega^*(t)f(t)f_j(t)\right]\right\}$$

$$E_2 = 2R(t)\ \frac{\partial R(t)}{\partial\theta_j}\left\{R(t)\frac{\partial f(t)}{\partial\theta_j} - f(t)\frac{\partial R(t)}{\partial\theta_j}\right\}$$

$$= -2\ R(t)p_j^2(t)F_j(t)\omega^*(t)\left\{R(t)f_j(t)k_j^*(t) + f(t)F_j(t)\omega^*(t)\right\},$$

where $f_j(t)$, $F_j(t)$, $k_j^*(t)$, $k_j^{*`}(t)$ and $\omega^*(t)$ are as given respectively, by (8), (9) (20), (75) and (21).

For i , j = 1 , 2 and i ≠ j ,

$$\emptyset_{ij} = \frac{\partial^2\emptyset}{\partial\theta_i\partial\theta_j} = \frac{E^*{}_1 - E^*{}_2}{(R(t))^4}, \qquad (80)$$

where

$$E_1^* = (R(t))^2\left\{\frac{\partial R(t)}{\partial\theta_i}.\frac{\partial f(t)}{\partial\theta_j} - \frac{\partial f(t)}{\partial\theta_i}.\frac{\partial R(t)}{\partial\theta_j}\right\}$$

$$= -\ (R(t))^2 p_i p_j \omega^*(t)\left\{F_i(t)f_j(t)k_j^*(t) - f_i(t)F_j(t)k_i^*(t)\right\}$$

$$E_2^* = 2R(t)\ \frac{\partial R(t)}{\partial\theta_i}\left\{R(t)\frac{\partial f(t)}{\partial\theta_j} - f(t)\frac{\partial R(t)}{\partial\theta_j}\right\}$$

$$= -\ 2R(t)p_i p_j F_i(t)\omega^*(t)\left\{R(t)f_j(t)k_j^*(t) + f(t)F_i(t)\omega^*(t)\right\},$$

Substituting (78 - 80) and (67 – 70) in (49) yields the Bayes estimator under squared error loss function, $\tilde{\tilde{H}}_s$, of H(t).

4.3. Approximate Bayes Estimation (Lindley's Procedure) Under LINEX Loss Function

On the basis of the LINEX loss function (47), the Bayes estimate of a function $w = w(\theta_1, \theta_2)$ of the unknown parameters θ_1 and θ_2 is given by

$$\hat{w}_L = -\frac{1}{c} \ln E(e^{-cw} \mid \underline{t}), \qquad c \neq 0, \tag{81}$$

where

$$E(e^{-cw}|t) = \frac{\int_\Omega e^{-cw(\underline{\theta})} L(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}{\int_\Omega L(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}}, \tag{82}$$

where Ω is the region in the $\theta_1$and θ_2 plane on which the posterior density $\pi(\underline{\theta}|t)$ is positive.

Suppose $\emptyset(\underline{\theta}) = e^{-cw(\underline{\theta})}$, we can apply Lindley's approximation to evaluate (49) so we obtain the following:

Bayes Estimation of the vector of parameters: The two parameters θ_1 and θ_2 can be approximately estimated by using Lindley's approximation from (49), as follow:

i. The Bayes estimator of the parameter $\boldsymbol{\theta_1}$:

Set $\emptyset(\underline{\theta}) = e^{-c\,\theta_1}$ in (2.12) then

$$\emptyset_1 = \frac{\partial \emptyset}{\partial \theta_1} = -ce^{-c\,\theta_1}, \ \emptyset_{11} = \frac{\partial^2 \emptyset}{\partial \theta_1^2} = c^2 e^{-c\,\theta_1}, \ \emptyset_2 = \frac{\partial \emptyset}{\partial \theta_2} = 0$$

$$\emptyset_{22} = \frac{\partial^2 \emptyset}{\partial \theta_2^2} = 0 \text{, for } i,j = 1{,}2 \ and \ i \neq j, \emptyset_{ij} = \frac{\partial^2 \emptyset}{\partial \theta_i \partial \theta_j} = 0$$

Substituting the above functions and (67 - 70) in (49) then into (81) yields the Bayes estimator under LINEX loss function, $\tilde{\theta}_{1,L}$ of θ_1

ii. The Bayes estimator of the parameter $\boldsymbol{\theta_2}$:

Set $\emptyset(\underline{\theta}) = e^{-c\,\theta_2}$ in (2.12), then

$$\emptyset_2 = \frac{\partial \emptyset}{\partial \theta_2} = -ce^{-c\,\theta_2}, \emptyset_{22} = \frac{\partial^2 \emptyset}{\partial \theta_2^2} = c^2 e^{-c\,\theta_2}, \emptyset_1 = \frac{\partial \emptyset}{\partial \theta_1} = 0,$$

$$\emptyset_{11} = \frac{\partial^2 \emptyset}{\partial \theta_1^2} = 0 \text{, for } i,j = 1{,}2 \ and \ i \neq j, \emptyset_{ij} = \frac{\partial^2 \emptyset}{\partial \theta_i \partial \theta_j} = 0$$

Substituting the above functions and (67 - 70) in (49) then into (81) yields the Bayes estimate under LINEX loss function , $\tilde{\tilde{\theta}}_{2,L}$ of θ_2.

The Bayes estimate of RF:

Set $\emptyset(\underline{\theta}) = e^{-c\,R(t)}$ in (49) where R(t) is given as in (15). Then for j = 1,2

$$\emptyset_j = \frac{\partial \emptyset}{\partial \theta_j} = \frac{\partial e^{-c\,R(t)}}{\partial \theta_j} = ce^{-c\,R(t)} p_j F_j(t) \omega^*(t), \tag{83}$$

for i, j =1,2 and $i \neq j$,

$$\emptyset_{ij} = \frac{\partial^2 \emptyset}{\partial \theta_i \partial \theta_j} = c^2 p_i p_j F_i(t) F_j(t) \omega^{*2}(t) e^{-c\,R(t)}, \tag{84}$$

and

$$\emptyset_{jj} = \frac{\partial^2 \emptyset}{\partial \theta_j^2} = \frac{\partial^2 e^{-c\,R(t)}}{\partial \theta_j^2} = c p_j \omega^*(t) e^{-c\,R(t)} \left[f_j(t) + c p_j \omega^*(t) F_j^2(t) \right] \qquad (85)$$

Where $f_j(t)\,, F_j(t)\ and\ \omega^*(t)$ are as given, respectively, by (8), (9) and (21)

Substituting (83 – 85) and (67 - 70) in (49) then into (81) yields the Bayes estimator under LINEX loss function, $\tilde{\tilde{\mathrm{R}}}_{\mathrm{L}}$ of $\mathrm{R(t)}$.

The Bayes estimator of HF: Set $\emptyset(\underline{\theta}) = e^{-c\,H(t)}$ in (49) where H(t) is given as in (16). Then for j = 1, 2

$$\emptyset_j = \frac{\partial \emptyset}{\partial \theta_j} = \frac{\partial e^{-c\,H(t)}}{\partial \theta_j} = -c e^{-c\,H(t)} \frac{\partial H(t)}{\partial \theta_j},$$

where $\frac{\partial H(t)}{\partial \theta_j}$ is defined in (4.2.29), then

$$\emptyset_j = \frac{-c p_j e^{-c\,H(t)}}{(R(t))^2} \delta_j^{\star}, \qquad (86)$$

where

$$\delta_j^{\star} = R(t) f_j(t) k_j^*(t) + f(t) F_j(t) \omega^*(t), \qquad (87)$$

$$\emptyset_{jj} = \frac{\partial^2 \emptyset}{\partial \theta_j^2} = \frac{\partial^2 e^{-c\,H(t)}}{\partial \theta_j^2}$$

$$= \frac{-c P_j e^{-c\,H(t)}}{(R(t))^4} \left[(R(t))^2 \left[f_j(t) \left(R(t) k_j^{*\grave{}}(t) + R(t) k_j^{*^2}(t) + \omega^*(t) f(t) \right) - \frac{c p_j {\delta_j^*}^2}{(R(t))^2} \right] + 2 p_j R(t) F_j(t) \omega^*(t) \delta_j^*(t) \right] \qquad (88)$$

for i , j = 1,2, and i≠j .

$$\emptyset_{ij} = \frac{\partial^2 \emptyset}{\partial \theta_i \partial \theta_j}$$

$$= \frac{-c p_i p_j e^{-c\,H(t)}}{(R(t))^4} \left\{ \left[\left(f_i(t) k_i^* F_j(t) - f_j(t) k_j^*(t) F_i(t) \right) \omega^*(t) - \frac{C \delta_i \delta_j}{(R(t))^2} \right] (R(t))^2 + 2 R(t) F_i(t) \omega^*(t) \delta_j(t) \right\}, \qquad (89)$$

where $f_j(t), F_j(t)\,, K_j^*(t)\,, K_j^{*\grave{}}(t)\,, \delta_j^*(t)\ and\ \omega^*(t)$ are given, respectively by (8), (9), (20), (75), (87) and (21).

Substituting $(86), (88), (89)$ and $(67 - 70)$ in (49) then in to (81), yields the Bayes estimation under LINEX loss function, $\tilde{\tilde{\mathrm{H}}}_{\mathrm{L}}$ of $\mathrm{H(t)}$.

5. Numerical illustration

5.1. Real Data Set

We obtained in the above chapters [3 and 4], Bayesian [Bayes and approximate Bayes] and non-Bayesian (MLE's) estimators of the vector parameters $\underline{\theta}$, of MTEW distribution. We adopted the squared error loss and LINEX loss function. In order to asses the statistical performances of these estimators, a real data study is conducted. The data set is from Cancho et al. (2007).

To illustrate the approaches developed in the previous chapters [3 and 4], we consider the data set presented in Aarset (1987) to identify the bathtub hazard rate contains life time of 50 industrial devices put on life test at time zero. Cancho et al. (2007) considered the EW distribution and developed a Bayesian methodology for analysis of lifetime data with hazard in form bathtub.

Considering the data in Aarset (1987), table (A) we fit (MTEW) distribution to the data set and summarized it in tables (1)and (2) by using MATHCAD package (2011).

Table (A): Life times for the 50 devices

T	**0.1**	**0.2**	**1**	**1**	**1**	**1**	**1**	**2**	**3**	**6**	**7**	**11**	**12**	**18**
t	**18**	**18**	**18**	**18**	**21**	**32**	**36**	**40**	**45**	**46**	**47**	**50**	**55**	**60**
t	**63**	**63**	**67**	**67**	**67**	**67**	**72**	**79**	**82**	**82**	**83**	**84**	**84**	
t	**84**	**85**	**85**	**85**	**85**	**85**	**86**	**86**						

For comparison purpose we compute Bayes and approximate for the parameters of the mixture of two exponentiated Weibull distribution when $\alpha_1 = 1,\ \theta_1 = 0.5,\ \alpha_2 = 1,\ \theta_2 = 0.5, p = 0.2, r = 6.$

Table (1): Bayesian estimates for the two shape parameters θ_1 and θ_2 of MTEW distribution for Aarset (1987) data

Parameters (θ_1, θ_2)	**SEL**			**LINEX cc=0.22**		
	Bayes	**MSE**	**Risk**	**Bayes**	**MSE**	**Risk**
$\theta_1 = 0.5$	0.570	0.200	0.195	0.510	0.004	0.003
$\theta_2 = 0.5$	0.590	0.200	0.192	0.510	0.003	0.002

Table (2): Approximate Bayesian (Lindley's approximation) estimates for the two shape parameters θ_1, and θ_2 of MTEW distribution for Aarset (1987) data

Parameters (θ_1, θ_2)	**SEL**			**LINEX cc=0.22**		
	Mean	**MSE**	**Risk**	**Mean**	**MSE**	**Risk**
$\theta_1 = 0.5$	0.550	0.200	0.198	0.560	0.030	0.030
$\theta_2 = 0.5$	0.490	0.010	0.009	0.500	0.001	0.001

Concluding Remarks:

In this research, we have presented the Bayes and approximate Bayes of the vector parameters $\underline{\theta}$ of MTEW distribution. The estimations are conducted on the basis of type-II censored samples. Bayes and approximate Bayes, under squared error loss and LINEX loss functions, are derived by using Lindely's method. The MLE's are also obtained. Our observations about the results are stated in the following points:

1. Table 1 shows the Bayes estimators under the squared error and LINEX loss functions, it indicated that the Bayes estimators under LINEX loss function are the best estimates as compared with the biases estimators under squared error loss function or MLE's. On the other hand, the Bayes estimator's under the LINEX loss function has the smallest estimated MSE's as compared with the estimates under squared error loss function.
2. Table 2 shows the approximate Bayes estimators under the squared error and LINEX loss functions it indicated that the Bayes estimators under squared error loss function are the best estimates as compared with the biases estimators under LINEX loss function or Bayes and MLE's. On the other hand, the approximate Bayes estimator's under the LINEX loss function has the smallest estimated MSE's as compared with the estimates under quadratic loss function or Bayes.

5.2. Simulation Study

In the above sections [3and 4], Bayesian estimators of the vector parameter $\underline{\theta}$ of MTEW distribution are presented. Approximate Bayes and Bayes estimates with squared error loss function are also obtained.

In order to assess the statistical performances of these estimates, a simulation study is conducted. The computations are carried out for censoring percentages of 60% for each sample size (n =10, 15, 20, 25, 30, 40 and 50). The mean square errors (MSE's) using generated random samples of different sizes are computed for each estimator.

Simulation Study for (Bayesian and approximate Bayesian) methods

Also, MATHCAD package was used to evaluate Bayes and approximate Bayes estimators under censored type-II using equations [(57) and (59)] for Bayes and [(48)] for approximate Bayes and for parameters values ($\theta_1 = 2.8$, $\theta_2 = 4$). The performance of the resulting estimators of the parameters has been considered in terms of the mean square error (MSE). The Simulation procedures will be described below:

Step 1: 1000 random samples of sizes 10, 15, 20, 25, 30, 40, and 50 were generated from MTEW model. If U has a uniform (0, 1) random number, then $x_{i,j} = px1_{i,j} + (1-p)x2_{i,j}$ where $x1_{i,j} = [-\ln(1-(u_{i,j})^{\frac{1}{\theta_1}}]^{\frac{1}{\alpha_1}}$, $x2_{i,j} = [-\ln[1-(u_{i,j})^{\frac{1}{\theta_2}}]]^{\frac{1}{\alpha_2}}$ follows MTEW model. The selected values for parameters are ($\theta_1 = 2.8$, $\theta_2 = 4$)

Step 2: Choose the number of failure *r*, we choose *r* to be less than the sample size n.

Step 3: Numerical integration method was used for solving the equations (58) and (59), to obtain Bayes estimators under squared error loss and LINEX loss function or the posterior mean and the mean square error (MSE).

Step 4: The approximate Bayes (Lindley's) estimators relative to squared error loss, $\underline{\tilde{\tilde{\theta}}_S}$ computed, using (49) after considering the appropriate changes according to section (4.1). Also, the approximate Bayes (Lindley's) estimators relative to LINEX loss $\underline{\tilde{\tilde{\theta}}_L}$ computed, using (49) after considering the appropriate changes according to sectionsr (4.2 and 4.3).

Table (3): Bayes Estimates, MSE and Risk for different sized samples when *p* = 0.2, $\theta_1 = 2.8$ **and** $\theta_2 = 4$ **using type -II censoring (** $r = 0.8n$ **) from MTEW distribution**

n	**Parameters** (θ_1, θ_2)	**SEL** $\boldsymbol{\theta_1 = 2.8}$, $\boldsymbol{\theta_2 = 4}$			**LINEX** $\boldsymbol{\theta_1 = 2.8}$, $\boldsymbol{\theta_2 = 4, c = 0.22}$		
		Bayes	**MSE**	**Risk**	**Bayes**	**MSE**	**Risk**
10	θ_1	1.659	1.364	0.063	1.883	0.875	0.035
	θ_2	2.830	1.423	0.054	3.037	0.974	0.047
15	θ_1	3.883	1.232	0.060	3.644	0.740	0.028
	θ_2	5.154	1.383	0.050	4.925	0.895	0.039
20	θ_1	1.755	1.150	0.058	2.104	0.503	0.019
	θ_2	2.908	1.243	0.051	3.129	0.784	0.026
25	θ_1	3.839	1.132	0.053	3.414	0.383	0.006
	θ_2	2.965	1.116	0.043	3.267	0.557	0.020
30	θ_1	3.819	1.085	0.045	3.391	0.352	0.003
	θ_2	3.018	1.006	0.041	4.603	0.389	0.025
35	θ_1	3.815	1.062	0.032	3.373	0.332	0.004
	θ_2	3.033	.9720	0.036	4.596	0.352	0.020
40	θ_1	1.787	1.052	0.026	2.252	0.303	0.003
	θ_2	3.096	0.848	0.031	3.441	0.324	0.012
45	θ_1	1.792	1.036	0.019	2.267	0.289	0.005
	θ_2	3.125	0.787	0.022	3.463	0.299	0.010
50	θ_1	1.832	0.959	0.021	2.311	0.243	0.003
	θ_2	3.219	0.629	0.019	3.493	0.263	0.006

Table (4): Approximate Bayes (Lindley's form) Estimates, MSE and Risk for different sized samples when p = 0.2, $\theta_1 = 2.8$ and $\theta_2 = 4$ using type -II censoring ($r = 0.8n$) from MTEW distribution

n	Parameters (θ_1, θ_2)	SEL $\theta_1 = 2.8$, $\theta_2 = 4$			LINEX $\theta_1 = 2.8$, $\theta_2 = 4, c = 0.22$		
		Lindley	MSE	Risk	Lindley	MSE	Risk
10	θ_1	1.718	1.174	0.022	3.449	0.431	0.009
	θ_2	2.971	1.073	0.014	3.219	0.614	0.004
15	θ_1	3.859	1.143	0.020	2.313	0.243	0.006
	θ_2	2.987	1.037	0.010	3.350	0.424	0.002
20	θ_1	1.755	1.112	0.020	2.364	0.195	0.005
	θ_2	3.000	1.009	0.009	4.580	0.342	0.005
25	θ_1	1.772	1.070	0.012	2.399	0.162	0.001
	θ_2	3.114	0.785	0.011	4.468	0.223	0.004
30	θ_1	1.789	1.039	0.017	3.146	0.121	0.001
	θ_2	3.359	0.416	0.005	3.577	0.182	0.003
35	θ_1	3.796	1.002	0.009	3.111	0.101	0.004
	θ_2	4.531	0.283	0.001	4.397	0.160	0.002
40	θ_1	1.943	0.740	0.005	2.507	0.089	0.003
	θ_2	3.624	0.146	0.005	3.671	0.109	0.001
45	θ_1	2.071	0.538	0.007	2.574	0.053	0.002
	θ_2	3.671	0.110	0.002	3.693	0.096	0.002
50	θ_1	2.306	0.246	0.002	2.952	0.026	0.003
	θ_2	3.732	0.073	0.001	3.765	0.063	0.008

Concluding Remarks:

Simulation results are displayed in tables 3 and 4, which give the posterior mean and MSE. Simulation studies are adopted for different sized samples. We have presented the Bayesian, and approximate Bayesian estimators of the vector parameters $\underline{\theta}$, of the life times follow MTEW distribution. The estimator is conducted on the basis of type- II censored samples. Bayes estimators, under squared error loss and LINEX loss functions, and Approximate Bayes form by using Lindley's method.

1. Our observations about the results are stated in the following Points: Table (3) shows the Bayes estimates under quadratic and LINEX loss functions. From table (3), we conclude that the Bayes estimates under LINEX loss function are the best estimates as compared with the biases of estimates under quadratic loss function. It is immediate to note that MSE's decrease as sample size n and the number of replication N increases. On the other hand Bayes estimates under the LINEX loss

function have the smallest estimated MSE's as compared with the estimates under quadratic loss function.

2. Table (4) shows the approximate Bayes (Lindley's) estimates under quadratic and LINEX loss functions. From table (4) we conclude that the Bayes estimates under the LINEX loss function are the best estimates as compared with the biases of estimates under quadratic loss function or Bayesian. It is immediate to note that MSE's decrease as sample size n and the number of replication (N) increases. On the other hand, Bayes estimates under the LINEX loss function have the smallest estimated MSE's as compared with the estimates under quadratic loss function or Bayesian.

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Appendix

The following proofs for some equations which were included in this research.

Proof of equation (67) and (68):

$$l_{30}^{*}=\frac{\partial_{111}}{\partial\theta_1}=-p_1\left\{\sum_{i=1}^{r}\frac{\partial A_1^{*}(t_{i:n})}{\partial\theta_1}+(n-r)\omega^{*}(t_{r:n})\frac{\partial B_1^{*}(t_{r:n})}{\partial\theta_1}\right\},$$

and

$$l_{03}^{*}=\frac{\partial^2 l_j}{\partial\theta_2^2}=-p_2\left\{\sum_{i=1}^{r}A_2^{*\backslash}(t_{i:n})+(n-r)\omega^{*}(t_{r:n})B_2^{*\backslash}(t_{r:n})\right\}$$

$$=-p_2\left\{\sum_{i=1}^{r}\frac{\partial A_2^{*}}{\partial\theta_2}+(n-r)\omega^{*}(t_{r:n})\frac{\partial B_2^{*}(t_{r:n})}{\partial\theta_2}\right\},$$

where

$$\frac{\partial A_j(t_{i:n})}{\partial\theta_j}=\theta_j^{-2}\xi_j^{**\backslash}(t_{i:n})-2\theta^{3}\xi_j^{**}(t_{i:n})-p_s\xi_s^{**}(t_{i:n})[k_j^{*2}\xi_j^{**\backslash}(t_{i:n})+2k_j^{*}(t_{i:n})k_j^{*\backslash}(t_{i:n})\xi_j^{**}(t_{i:n})],$$

and $$\frac{\partial B_j^{*}(t_{r:n})}{\partial\theta_j}=\omega^{*}(t_{r:n})\xi_j^{***\backslash}(t_{r:n})+2p_j\omega^{*}(t_{r:n})\xi_j^{***}(t_{r:n})\xi_j^{***\backslash}(t_{r:n}),$$

Proof of equations (69) and (70):

$$l_{21}^{*}=\frac{\partial l_{21}}{\partial\theta_1}=-p_1p_2\left\{\sum_{i=1}^{r}\frac{\partial\varphi^{*}(t_{i:n})}{\partial\theta_1}+(n-r)\omega^{*2}(t_{r:n})\frac{\partial\Psi^{*}(t_{r:n})}{\partial\theta_1}\right\},$$

$$l_{12}^{*}=\frac{\partial l_{12}}{\partial\theta_2}=-p_1p_2\left\{\sum_{i=1}^{r}\frac{\partial\varphi^{*}(t_{i:n})}{\partial\theta_2}+(n-r)\omega^{*2}(t_{r:n})\frac{\partial\Psi^{*}(t_{r:n})}{\partial\theta_2}\right\},$$

Where, $\frac{\partial\varphi^{*}(t_{i:n})}{\partial\theta_j}=k_s^{*}(t_{i:n})\xi_s^{**}(t_{i:n})[k_j^{*}(t_{i:n})\xi_j^{**\backslash}(t_{i:n})+\xi_j^{**}(t_{i:n})k_j^{*\backslash}(t_{i:n})],$

$$\frac{\partial\Psi^{*}(t_{r:n})}{\partial\theta_j}=\xi_s^{***}(t_{r;n})\xi_j^{***\backslash}(t_{r;n}).$$