Concomitants of Generalized Order Statistics for a Bivariate Exponential Distribution

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Abstract

In this paper we have obtained the distribution of concomitant of order statistics for Bivariate Pseudo–Exponential distribution. The expression for moments has also been obtained. We have found that only the fractional moments of the distribution exist.

Keywords: Concomitants, Generalized Order Statistics, Exponential Distribution.

1. Introduction

The Generalized Order Statistics (GOS) has been introduced by Kamps (1995) as a unified model for ordered random variables. This model provide several models of ordered random variables as special case. The GOS are denoted as $X_{r:n,m,k}$ and Kamps (1995) has argued that the quantities $X_{r:n,m,k}$ are called GOS if their joint distribution is given as:

$$f_{1,\ldots,n:n,m,k}\left(x_1,x_2,\ldots,x_n\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{1-F\left(x_n\right)\right\}^{k-1}f\left(x_n\right)$$
$$\times\left[\prod_{j=1}^{n-1}\left\{1-F\left(x_i\right)^m\right\}f\left(x_i\right)\right]; \quad (1.1)$$

where *n* is sample size, *m* and *k* are parameters of the model and quantities γ_j are given as $\gamma_j=k+\left(n-r\right)\left(m+1\right)$. Kamps (1995) has shown that the density function of *r*th GOS is given as

$$f_{r:n,m,k}\left(x\right)=\frac{C_{r-1}}{\left(r-1\right)!}f\left(x\right)\left\{1-F\left(x\right)\right\}^{\gamma_r-1}g_m^{r-1}\left[F\left(x\right)\right], \quad (1.2)$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_j;r=1,2,\ldots,n$, and

$$g_m\left(x\right)=h_m\left(x\right)-h_m\left(0\right)=\begin{cases}\left[1-\left(1-x\right)^{m+1}\right]\Big/\left(m+1\right); & m\neq-1\\ -\ln\left(1-x\right) & m=-1.\end{cases}$$

We also have

$$h_m\left(x\right)=\begin{cases}-\left(1-x\right)^{m+1}\Big/\left(m+1\right); & m\neq-1\\ -\ln\left(1-x\right) & m=-1.\end{cases}$$

Kamps (1995) has further shown that the joint density function of two GOS $X_{r:n,m,k}$ and $X_{s:n,m,k}$ for $r<s$ is given as

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times\{1-F(x_2)\}^{\gamma_s-1}\left[h_m\{F(x_2)\}-h_m\{F(x_1)\}\right]^{s-r-1};-\infty<x_1<x_2<\infty. \quad (1.3)$$

The density functions of GOS given in (1.2) and (1.3) provide several models of ordered random variables as special case. Specifically for $m=0$ and $k=1$ the model reduces to *Ordinary Order Statistics* as given by David and Nagaraja (2003). Also for $m=-1$ we obtain *k*th upper record values introduced by Chandler (1952). Other models like fractional order statistics given by Stigler (1977), sequential order statistics etc. can also be obtained for various values of the parameters involved.

Often it happen that we have a sample from a bivariate distribution and the sample is arranged with respect to one of the variable. The automatically shuffled variable is referred to as the concomitants of ordered random variables. For illustration if (X_1,Y_1), (X_2,Y_2), …, (X_n,Y_n) be a sample from some bivariate distribution and sample is arranged with respect to variable *X*, then the variable *Y* is called concomitant of ordered random variable.

Concomitants of order statistics have been discussed by David and Nagaraja (2003) and the concomitants of record values have been discussed by Ahsanullah (1995). Several authors have studied the concomitants of order and record statistics. Some notable references include Nagaraja and David (1994), Wang et al. (2006), Shahbaz et al. (2009), Ahsanullah et al. (2010), Mohsin et al. (2010), Shahbaz and Shahbaz (2010) among others.

The concept of concomitants of order and record statistics has been extended to the concomitants of GOS by Ahsanullah and Nevzorov (2001). Specifically, the distribution of concomitant of *r*th GOS is given as

$$f_{[r:n,m,k]}(y)=\int_{-\infty}^{\infty}f(y|x)f_{r:n,m,k}(x)dx, \quad (1.4)$$

where $f(y|x)$ is conditional distribution of *Y* given $X = x$ and $f_{r:n,m,k}(x)$ is defined in (1.2). The joint distribution of two concomitants is given as

$$f_{[r,s:n,m,k]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f(y_1|x_1)f(y_2|x_2)f_{r,s:n,m,k}(x_1,x_2)dx_1dx_2, \quad (1.5)$$

where $f_{r,s:n,m,k}(x_1,x_2)$ is given in (1.3).

Concomitants of GOS has not been studied much extensively. Ahsanullah and Beg (2006) have studied concomitants of GOS for Gumbel Bivariate Exponential distribution. Further Beg and Ahsanullah (2008) has studied concomitants of GOS for Gumbel bivariate family of distributions. Nayabuddin (2013) has studied concomitants of GOS for bivariate Lomax distribution.

In this paper we will study concomitants of GOS for a bivariate Exponential distribution introduced by Mohsin et al. (2010). In section 2 we give a brief review of bivariate Exponential distribution introduced by Mohsin et al. (2013). Concomitants of single GOS for bivariate Exponential distribution is discussed in section 3 with some properties of the distribution. The joint distribution of concomitants is discussed in section 4.

2. A Bivariate Exponential Distribution

Fillus and Fillus (2006) introduced a new method of generating multivariate distributions by using the given marginal. They have introduced Pseudo–Weibull and Pseudo–Gamma distributions as linear combinations of random variables. Shahbaz and Shahbaz (2010) and Mohsin et al. (2010) have introduced the Pseudo-Exponential distribution as a compound distribution of two random variables as under:

Suppose a random variable X has an exponential distribution with parameter α. The density function of X is:

$$f\left(x;\alpha\right)=\alpha e^{-\alpha x},\alpha>0,x>0. \qquad (2.1)$$

Suppose further that the random variable Y also has the exponential distribution with parameter $\phi\left(x\right)$, where $\phi\left(x\right)$ is some function of random variable X. The density function of Y is; therefore:

$$f\left\{y;\phi\left(x\right)\middle|\,x\right\}=\phi\left(x\right)e^{-\phi\left(x\right)y}\ ;\ \phi\left(x\right)>0\ ,\ y>0. \qquad (2.2)$$

The compound distribution of (2.1) and (2.2) has been called the Bivariate Pseudo–Exponential distribution by Shahbaz et al. (2009) and by Mohsin et al. (2010). The density function of this distribution is:

$$f\left(x,y\right)=\alpha\phi\left(x\right)exp\left[-\left\{\alpha x+\phi\left(x\right)y\right\}\right];\ \ \alpha>0\ ,\phi\left(x\right)>0\ ,\ x>0\ ,\ y>0. \qquad (2.3)$$

Various choices of $\phi\left(x\right)$ in (2.3) will lead to a different bivariate distributions. Shahbaz et al. (2009) have used $\phi\left(x\right)=x$ in (2.3) to obtain the following bivariate distribution:

$$f\left(x,y\right)=\alpha xexp\left[-x\left\{\alpha+y\right\}\right];\quad \alpha>0\ ,\ x>0\ ,\ y>0. \qquad (2.4)$$

The product moments of (2.4) are given as:

$$\mu_{p,q}^{/}=\int_0^\infty\int_0^\infty x^p y^q f\left(x,y\right)dxdy=\int_0^\infty\int_0^\infty x^p y^q\alpha xexp\left[-x\left(\alpha+y\right)\right]dxdy;$$

which after simplification becomes:

$$\mu_{p,q}^{/}=\alpha^{q-p}\Gamma\left(p-q+1\right)\Gamma\left(q+1\right); \qquad (2.5)$$

The product moments exist for $q < p+1$. Shahbaz et al. (2009) have studied the concomitants of order statistics for distribution (2.4) and Mohsin et al. (2010) have studied the concomitants of records for the same distribution.

In the following section we have obtained the distribution of concomitant of GOS for (2.4).

3. Distribution of r–th Concomitant of GOS

The Bivariate Exponential distribution introduced by Shahbaz et al. (2009) is given in (2.4). The distribution of $r-th$ concomitant for (2.4) can be obtained by using (1.4). The distribution is obtained by first obtaining the distribution of $f_{r:n,m,k}(x)$. For this we first see that the marginal distribution of *X* is given in (2.1). We also have

$$F(x) = 1 - e^{-\alpha x}.$$

The distribution of $f_{r:n,m,k}(x)$ is given in (1.2). Now for the distribution (2.1) we have:

$$g_m\left[F(x)\right] = \frac{1}{m+1}\left[1-\exp\left\{-\alpha(m+1)x\right\}\right]$$

$$g_m^{r-1}\left[F(x)\right] = \frac{1}{(m+1)^{r-1}}\left[1-\exp\left\{-\alpha(m+1)x\right\}\right]^{r-1}$$

$$= \frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\exp\left\{-\alpha(m+1)ix\right\}. \quad (3.1)$$

Now using (2.1) and (3.1) in (1.2) the distribution of $f_{r:n,m,k}(x)$ is given as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}\alpha\exp(-\alpha x)\exp\left\{-\alpha x(\gamma_r - 1)\right\}$$

$$\times\frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\exp\left\{-\alpha(m+1)ix\right\}$$

or $$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\alpha\exp(-\alpha w_1 x), \quad (3.2)$$

where $w_1 = \left\{(m+1)i + \gamma_r\right\}$.

We also see that the conditional distribution of *Y* given *X* for distribution (2.4) is

$$f(y|x) = xe^{-xy}; x, y > 0. \quad (3.3)$$

Using (3.2) and (3.3) in (1.4), the distribution of *r*th concomitant of GOS is obtained as below:

$$f_{[r:n,m,k]}(y) = \int_0^\infty y\exp(-xy)\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\alpha\exp(-\alpha w_1 x)dx$$

or $$f_{[r:n,m,k]}(y)=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\alpha}{(y+\alpha w_1)^2};y>0. \qquad (3.4)$$

The distribution (3.4) reduces to the distribution of concomitants for *r*th Order Statistics for *m=0* and *k=1* as obtained by Shahbaz et al. (2009). Further (3.4) reduces to the distribution of concomitants of record statistics for *m*=–1 as obtained by Mohsin et al. (2010).

The expression for *p*th moment of *r*th GOS is readily obtained from (3.4) as

$$\mu^{p}_{[r:n,m,k]}=\int_0^\infty y^p f_{[r:n,m,k]}(y)dy=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^y\frac{\alpha\beta_2 y^{\beta_2-1}}{(y^{\beta_2}+\alpha w_1)^2}dy$$

or $$\mu^{p}_{[r:n,m,k]}=\frac{\alpha pC_{r-1}\Gamma(p)\Gamma(1-p)}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}(\alpha w_1)^{p-1}, \qquad (3.5)$$

which exist for $|p|<1$. We can see that the moment expression given in (3.5) reduces to expression for moments of concomitants of order statistics given by Shahbaz et al. (2009) for *m=0* and *k=1*.

The moments can be computed numerically for various choices of the parameters involved. Shahbaz et al. (2009) has computed numerical values for $p=0.5$ and n = 10. The same can be obtained from (3.5) by using $n=10, p=0.5, m=0$ and k = 1.

The distribution function of *r*th concomitant of GOS is readily written from (3.4) as

$$F_{[r:n,m,k]}(y)=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\alpha y}{w_1(y+\alpha w_1)};y>0. \qquad (3.6)$$

Using (3.4) and (3.6), the survivorship function is immediately written as

$$S(t)=\frac{\dfrac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\dfrac{\alpha}{(t+\alpha w_1)^2}}{1-\dfrac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\dfrac{\alpha t}{w_1(t+\alpha w_1)}}. \qquad (3.7)$$

The survivorship function can be computed for various choices of the parameters.

The joint distribution of two concomitants of GOS is obtained in the following section.

4. Joint Distribution of Two Concomitants of GOS

The joint distribution of concomitants of GOS is given in (1.5) as

$$f_{[r,s:n,m,k]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f(y_1|x_1)f(y_2|x_2)f_{r,s:n,m,k}(x_1,x_2)dx_1dx_2,$$

where $f_{r,s:n,m,k}(x_1,x_2)$ is given in (1.3). Now for given distribution we have

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\alpha\exp(-\alpha x_1)\alpha\exp(-\alpha x_2)\exp(-\alpha m x_1)$$
$$\times\frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\exp\{-\alpha(m+1)ix_1\}\exp\{-\alpha x_1(\gamma_s-1)\}\frac{1}{(m+1)^{s-r-1}}$$
$$\times\left[\exp\{-\alpha(m+1)x_1\}-\exp\{-\alpha(m+1)x_2\}\right]^{s-r-1};$$

which after simplification becomes

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\alpha^2\exp(-\alpha w_2x_1)\exp(-\alpha w_3x_2);\qquad 0<x_1<x_2<\infty, \tag{4.1}$$

where $w_2=\{(m+1)(s-r-j+i)\}$; $w_3=\{(m+1)j+\gamma_s\}$.

Using (4.1) and (3.3) in (1.5) we have the joint distribution of two concomitants of GOS as

$$f_{[r,s:n,m,k]}(y_1,y_2)=\int_0^\infty\int_{x_1}^\infty x_1\exp(-x_1y_1)x_2\exp(-x_2y_2)\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\alpha^2\exp(-\alpha w_2x_1)\exp(-\alpha w_3x_2)dx_2dx_1$$

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\alpha^2\int_0^\infty x_1\exp\{-x_1(y_1+\alpha w_2)\}I(x_2)dx_1$$

where $I(x_2)=\int_{x_1}^\infty x_2\exp\{-x_2(y_2+\alpha w_3)\}dx_2$.

Now making the transformation $w=x_2(y_2+\alpha w_3)$ we have

$$I(x_2)=\frac{1}{(y_2+\alpha w_3)^2}\{1+(y_2+\alpha w_3)x_1\}\exp\{-x_1(y_2+\alpha w_3)\}.$$

So we have

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\alpha^2\frac{1}{(y_2+\alpha w_3)^2}$$
$$\times\int_0^\infty x_1\exp\{-x_1(y_1+\alpha w_2)\}\{1+(y_2+\alpha w_3)x_1\}\exp\{-x_1(y_2+\alpha w_3)\}dx_1$$

Simplifying above, the joint density of two concomitants of GOS for bivariate Exponential distribution is given as:

$$f_{[r,s:n,m,k]}\left(y_1,y_2\right)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\alpha^2\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}$$
$$\times\binom{s-r-1}{j}\frac{\left\{y_1+3y_2+\alpha\left(w_2+w_3\right)\right\}}{\left(y_2+\alpha w_3\right)^2\left\{y_1+y_2+\alpha\left(w_2+w_3\right)\right\}^3}. \tag{4.2}$$

The product moments can be obtained from (4.2).

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