Certain Characterizations of Recently Introduced Distributions

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Abstract

Various characterizations of twenty four recently introduced distributions are presented. These characterizations are based on: (i) ratio of two truncated moments; (ii) the hazard function and (iii) conditional expectations.

Keywords: Univariate continuous distributions; Characterizations; Conditional expectations; Truncated moments.

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions.

The present work deals with certain characterizations of each of the distributions: Harris-G class (H-GC) of Pinho et al. (2015); Harris extended Burr XII (HEBXII); Harris extended exponentiated exponential (HEEE) (both) of Jose et al. (2015); exponentiated generalized Weibull Gompertz (EGWG) of El-Damcese et al. (2015); the beta exponentiated Lomax (BEL) distribution of Mead (2016) ; Kies (K); three-parameter extension of exponential (TPEE) of Lemonte et al. (2016); modified exponential-geometric (MEG) of Bordbar et al. (2016); Kumarasawamy flexible Weibull extension (KFWE) of El-Damcese et al. (2016); transmuted exponentiated Pareto-I (TEP-I) of Fatima et al. (2016); transmuted Gompertz (TG) of Abdul-Moniem et al. (2015); the new extended Burr XII (EBXII) of Ghosh et al. (2016); Weibull Fréchet (WFr) of Afify et al. (2016); Marshall-Olkin gamma-Weibull (MOP $_{g}$W $_{\alpha}$) of Saboor et al. (2016); transmuted exponentiated Weibull geometric (TEWG) of Saboor et al. (2016); transmuted generalized Gompertz (TGG) of Khan et al. (2016); negative binomial Birnbaum-Saunders (NBBS) of Cordeiro et al. (2016); Marshall-Olkin extended generalized Rayleigh (MOEGR) of MirMostafaee et al. (2016); generalized inverse Lindley of Sharma et al. (2015); Kumaraswamy transmuted exponentiated additive Weibull (Kw-TEAW) of Nofal et al. (2016); beta exponentiated gamma (BEG $_{1}$) of Feroze et al. (2016); Kumaraswamy Kumaraswamy Weibull (KwKwW) of Mahmoud et al. (2016);

beta exponentiated Gumbel (BEG_2) of Ownuk (2015) and exponential Poisson Logarithmic (EPL) of Fioruci et al. (2016). These characterizations are based on: (i) ratio of two truncated moments; (ii) the hazard function and (iii) conditional expectations.

For detailed treatments and the domain of applicability of each of these distributions, we refer the interested reader to the corresponding papers cited in the References section.

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original authors.

A) The cdf and pdf of (H-GC) are given, respectively, by

$$F(x) = 1 - \left[\frac{\theta \overline{G}(x)^k}{1 - \overline{\theta}\overline{G}(x)^k}\right]^{1/k}, \qquad x \geq 0, \tag{1.1}$$

and

$$f(x) = \frac{\theta^{1/k} g(x)}{\left[1 - \overline{\theta}\overline{G}(x)^k\right]^{1+\frac{1}{k}}}, \qquad x > 0, \tag{1.2}$$

where, $k > 0$, $\theta \in (0,1)$, $\overline{\theta} = 1 - \theta$ are parameters and $G(x)$ $\left(\overline{G}(x) = 1 - G(x)\right)$ is the baseline cdf with the corresponding pdf $g(x)$.

B) The cdf and pdf of (HEBXII) are given, respectively, by

$$F(x) = 1 - \left[\frac{\theta (1 + x^c)^{-kb}}{1 - \overline{\theta}(1 + x^c)^{-kb}}\right]^{1/k}, \qquad x \geq 0, \tag{1.3}$$

and

$$f(x) = \frac{\theta^{1/k} b c x^{c-1} (1 + x^c)^{-(b+1)}}{\left[1 - \overline{\theta}(1 + x^c)^{-kb}\right]^{1+\frac{1}{k}}}, \qquad x > 0, \tag{1.4}$$

where, b, c, k and θ are all positive parameters.

C) The cdf and pdf of (HEEE) are given, respectively, by

$$F(x) = 1 - \frac{\theta^{1/k}\left[1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right]}{\left\{1 - \overline{\theta}[1 - (1 - e^{-\lambda x})^{\alpha}]^k\right\}^{1/k}}, \qquad x \geq 0, \tag{1.5}$$

and

$$f(x) = \frac{\theta^{1/k} \alpha \lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha - 1}}{\left\{1 - \overline{\theta}[1 - (1 - e^{-\lambda x})^{\alpha}]^k\right\}^{1+\frac{1}{k}}}, \qquad x > 0, \tag{1.6}$$

where, α, λ, k and θ are all positive parameters.

D) The cdf and pdf of (EGWG) are given, respectively, by

$$F(x) = \left[1 - \exp\left\{-\left[a x^b \left(e^{c x^d} - 1\right)\right]\right\}\right]^{\theta}, \qquad x \geq 0, \tag{1.7}$$

and

$$f(x) = ab\theta x^{b-1} e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\times$$
$$\left[1-\exp\left\{-\left[ax^b\left(e^{cx^d}-1\right)\right]\right\}\right]^{\theta}, \tag{1.8}$$

$x > 0$, where a, b, c, d and θ are all positive parameters.

Note: A special case of (EGWG) for $\theta = 1$, was taken up by El-Bassiouny et al. (2015) and characterized based on the upper record values. Another special case of (EGWG) for $\theta = d = 1$ and $b = 0$ has appeared in a paper by Maiti and Pramanik (2015).

E) The cdf and pdf of (BEL) distribution are given, respectively, by

$$F(x) = \frac{1}{B(a,b)}\int_0^{\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}} v^{a-1}(1-v)^{b-1}dv, \quad x \geq 0, \tag{1.9}$$

and

$$f(x) = \frac{\beta\theta\lambda}{B(a,b)}(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{a\beta-1}\times$$
$$\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}\right\}^{b-1}, \quad x > 0 \tag{1.10}$$

where $\beta, \theta, \lambda, a, b$ are all positive parameters and $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$.

F) The cdf and pdf of (K) are given, respectively, by

$$F(x) = 1-\exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}, \qquad a \leq x \leq b, \tag{1.11}$$

and

$$f(x) = \frac{\lambda\beta(b-a)(x-a)^{\beta-1}\exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}}{(b-x)^{\beta+1}}, \qquad a < x < b, \tag{1.12}$$

where $a, b\ (a < b), \beta$ and λ are all positive parameters.

G) The cdf and pdf of (TPEE) are given, respectively, by

$$F(x) = 1-\frac{\beta\exp\{1-(1+\lambda x)^{\alpha}\}}{1-(1-\beta)\exp\{1-(1+\lambda x)^{\alpha}\}}, \quad x \geq 0, \tag{1.13}$$

and

$$f(x) = \frac{\alpha\beta\lambda(1+\lambda x)^{\alpha-1}\exp\{1-(1+\lambda x)^{\alpha}\}}{\left\{1-(1-\beta)\exp\{1-(1+\lambda x)^{\alpha}\}\right\}^2}, \quad x > 0, \tag{1.14}$$

where α, β and λ are all positive parameters.

H) The cdf and pdf of (MEG) are given, respectively, by

$$F(x) = 1-\frac{\alpha(1-p)e^{-\beta x}}{1-\overline{\alpha}e^{-\lambda x}-\alpha p e^{-\beta x}}, \quad x \geq 0, \tag{1.15}$$

and

$$f(x) = \frac{\alpha\beta(1-p)e^{-\beta x}}{(1-\overline{\alpha}e^{-\lambda x}-\alpha p e^{-\beta x})^2}, \quad x > 0, \tag{1.16}$$

where α, β, λ all positive and $p\ (0 < p < 1)$ are parameters.

I) The cdf and pdf of (KFWE) are given, respectively, by

$$F(x) = 1 - \left\{1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a}\right\}^{b}, \quad x \geq 0, \tag{1.17}$$

and

$$f(x) = ab\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{-e^{\alpha x - \frac{\beta}{x}}} \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a-1} \times$$

$$\left\{1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^{a}\right\}^{b-1}, \qquad x > 0, \tag{1.18}$$

where α, β, a, b are all positive parameters.

Note: A special case of (KFWE) for $b = 1$, was taken up by El-Gohary et al. (2015).

J) The cdf and pdf of (TEP-I) are given, respectively, by

$$F(x) = 1 - k^a e^{-ax}[1 - \lambda(1 - k^a e^{-ax})], \quad x \geq \ln k, \tag{1.19}$$

and

$$f(x) = ak^a e^{-ax}[1 - \lambda(1 - 2k^a e^{-ax})], \quad x > lnk, \tag{1.20}$$

where $a, k > 0$ and $|\lambda| \leq 1$ are parameters.

K) The cdf and pdf of (TG) are given, respectively, by

$$F(x) = 1 - e^{-\theta(e^{\alpha x} - 1)}\left[1 - \lambda + \lambda e^{-\theta(e^{\alpha x} - 1)}\right], \quad x \geq 0, \tag{1.21}$$

and

$$f(x) = \alpha\theta e^{\alpha x} e^{-\theta(e^{\alpha x} - 1)}\left[1 - \lambda + 2\lambda e^{-\theta(e^{\alpha x} - 1)}\right], \quad x > 0, \tag{1.22}$$

where $\alpha, \theta > 0$ and $|\lambda| \leq 1$ are parameters.

L) The cdf and pdf of (EBXII) are given, respectively, by

$$F(x) = 1 - \frac{2}{1 + (1 + x^c)^{\lambda}}, \quad x \geq 0, \tag{1.23}$$

and

$$f(x) = \frac{2\lambda c x^{c-1}(1 + x^c)^{\lambda - 1}}{[1 + (1 + x^c)^{\lambda}]^2}, \quad x > 0, \tag{1.24}$$

where $\lambda, c > 0$ are parameters.

M) The cdf and pdf of (WFr) are given, respectively, by

$$F(x) = 1 - \exp\left\{-a\left(e^{(\alpha/x)^{\beta}} - 1\right)^{-b}\right\}, \quad x \geq 0, \tag{1.25}$$

and

$$f(x) = \frac{ab\beta\alpha^{\beta} x^{-\beta-1} e^{-b(\alpha/x)^{\beta}}}{\left\{1 - e^{-(\alpha/x)^{\beta}}\right\}^{b+1}} \exp\left\{-a\left(e^{(\alpha/x)^{\beta}} - 1\right)^{-b}\right\}, \quad x > 0, \tag{1.26}$$

where α, β, a, b are all positive parameters.

N) The cdf and pdf of (MOP $_g$W $_\alpha$) are given, respectively, by

$$F(x) = \frac{1 - Q\left(1 + \frac{\xi}{k}, \frac{x^k}{\lambda^k}\right)}{1 - (1-\alpha) Q\left(1 + \frac{\xi}{k}, \frac{x^k}{\lambda^k}\right)}, \quad x \geq 0, \tag{1.27}$$

and

$$f(x) = \frac{k\alpha\lambda^{-k-\xi} e^{-\lambda^{-k}x^k} x^{-1+\xi+k}}{\Gamma\left(1 + \frac{\xi}{k}\right)\left[1 - (1-\alpha) Q\left(1 + \frac{\xi}{k}, \frac{x^k}{\lambda^k}\right)\right]^2}, \quad x > 0, \tag{1.28}$$

where $\alpha > 0, k > 0$, $\xi + k > 0$ are parameters and $Q(a,z) = \frac{1}{\Gamma(a)} \int_x^\infty t^a e^{-t} dt$, $\Re(a) > 0$.

O) The cdf and pdf of (TEWG) are given, respectively, by

$$F(x) = \frac{(1-p)\left(1 - e^{-(\alpha x)^\beta}\right)^\theta}{1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta} \times \left\{1 + \lambda - \lambda\left[\frac{(1-p)\left(1 - e^{-(\alpha x)^\beta}\right)^\theta}{1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta}\right]\right\}, \quad x \geq 0, \tag{1.29}$$

and

$$f(x) = \theta\beta\alpha^\beta(1-p)x^{\beta-1}e^{-(\alpha x)^\beta} \times \left(1 - e^{-(\alpha x)^\beta}\right)^{\theta-1}\left[1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta\right]^{-2} \times \left\{1 + \lambda - 2\lambda\left[\frac{(1-p)\left(1 - e^{-(\alpha x)^\beta}\right)^\theta}{1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta}\right]\right\}, \quad x > 0, \tag{1.30}$$

where $p \in [0,1), \alpha, \beta, \theta > 0$, $|\lambda| \leq 1$ are parameters.

P) The cdf and pdf of (TGG) are given, respectively, by

$$F(x) = \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^\beta \times \left\{1 + \lambda - \lambda\left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^\beta\right\}, \quad x \geq 0, \tag{1.31}$$

and

$$f(x) = \alpha\beta e^{\xi x} \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta-1} \times \left\{1 + \lambda - 2\lambda\left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^\beta\right\}, \quad x > 0, \tag{1.32}$$

where $\alpha, \beta, \xi > 0$, $|\lambda| \leq 1$ are parameters.

Note: For $\beta = 1$, TGG reduces to TG (Transmuted Gompertz) Distribution, discussed by the same authors, which appeared in Pakistan Journal of Statistics, Vol. 32, No.3 , 2016.

Q) The cdf and pdf of (NBBS) are given, respectively, by

$$F(x) = \frac{1 - [1 + \theta\xi F_{BS}(x)]^{-1/\xi}}{1 - (1 + \theta\xi)^{-1/\xi}}, \quad x \geq 0, \tag{1.33}$$

and

$$f(x)=\frac{\theta f_{BS}(x)[1+\theta\xi F_{BS}(x)]^{-\left(\frac{1}{\xi}+1\right)}}{1-(1+\theta\xi)^{-1/\xi}},\qquad x>0, \tag{1.34}$$

where $\theta,\xi>0$ are parameters, and F_{BS} , f_{BS} are cdf and pdf of the Birnbaum-Saunders distribution. $F_{BS}(x)=\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\lambda}}-\sqrt{\frac{\lambda}{x}}\right)\right]$,where Φ is cdf of the standard normal and $\lambda>0$ is a parameter.

R) The cdf and pdf of (MOEGR) are given, respectively, by

$$F(x)=1-\frac{\alpha\overline{\gamma}(\lambda+1,\theta x^2)}{1-\overline{\alpha\gamma}(\lambda+1,\theta x^2)},\qquad x\ge 0, \tag{1.35}$$

and

$$f(x)=\frac{2\alpha\theta^{\lambda+1}x^{2\lambda+1}e^{-\theta x^2}}{\Gamma(\lambda+1)[1-\overline{\alpha\gamma}(\lambda+1,\theta x^2)]^2},\qquad x>0, \tag{1.36}$$

where $\alpha,\theta>0,\lambda>-1$ are parameters and $\overline{\gamma}(a,x)=1-\gamma(a,x)=1-\int_0^x t^{a-1}e^{-t}dt$.

S) The cdf and pdf of (GIL) are given, respectively, by

$$F(x)=\left(1+\frac{\theta}{(1+\theta)x^{\alpha}}\right)e^{-\theta x^{-\alpha}},\qquad x\ge 0, \tag{1.37}$$

and

$$f(x)=\frac{\theta x^2}{(1+\theta)}\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\theta x^{-\alpha}}\qquad x>0, \tag{1.38}$$

where $\alpha,\theta>0$ are parameters.

T) The cdf and pdf of (Kw-TEAW) are given, respectively, by

$$F(x)=1-\left\{\begin{matrix}1-\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{a\delta}\times\\ \left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\end{matrix}\right\}^{b},\qquad x\ge 0, \tag{1.39}$$

and

$$\begin{aligned}f(x)&=ab\delta e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\left[1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right]^{a\delta-1}\times\\ &\left\{1-\left[1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right]^{a\delta}\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\right\}^{b-1}\times\\ &\left\{1+\lambda-2\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right\}\times\\ &\left\{1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right\}^{a-1},\end{aligned} \tag{1.40}$$

$x>0$, where $\alpha,\beta,\gamma,\theta\ge 0$ with $\theta<1<\beta$ (or $\beta<1<\theta$) and $|\lambda|\le 1$ are parameters.

U) The cdf and pdf of (BEG_1) are given, respectively, by

$$F(x)=\frac{1}{B(a,b)}\int_0^{\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta}} w^{a-1}(1-w)^{b-1}\mathrm{d}w,\qquad x\ge 0, \tag{1.41}$$

and

$$f(x)=\frac{\theta\lambda^{2}xe^{-\lambda x}}{B(a,b)}\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a-1}\times$$
$$\left\{1-\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta}\right\}^{b-1},\quad x>0, \tag{1.42}$$

where a, b, λ and θ are all positive parameters.

V) The cdf and pdf of (KwKwW) are given, respectively, by

$$F(x)=1-\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]\right\}^{b},\quad x\geq 0, \tag{1.43}$$

and

$$f(x)=ab\alpha\beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta-1}$$
$$\times\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a-1}\times$$
$$\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1}, \tag{1.44}$$

$x>0$, where $a, b, \alpha, \beta, \theta$ and λ are all positive parameters.

W) The cdf and pdf of (BEG_2) are given, respectively, by

$$F(x)=\frac{1}{B(a,b)}\int_{0}^{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}}w^{a-1}(1-w)^{b-1}dw,\quad x\geq 0, \tag{1.45}$$

and

$$f(x)=\frac{\alpha\exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma B(a,b)}\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\times$$
$$\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha b-1}\times$$
$$\left\{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right\}^{a-1},\quad x>0, \tag{1.46}$$

where α, σ, a, b all positive, $\mu\in\mathbb{R}$ are parameters and $B(a,b)=\int_{0}^{1}u^{a-1}(1-u)^{b-1}du$.

X) The cdf and pdf of (EPL) are give, respectively, by

$$F(x)=1-\frac{\log\left[1-\phi\left(\frac{1-\exp\left(-\theta e^{-\lambda x}\right)}{1-e^{-\theta}}\right)\right]}{1-e^{-\theta}},\quad x\geq 0, \tag{1.47}$$

and

$$f(x)=\frac{\phi\theta\lambda\exp\left(-\lambda x-\theta e^{-\lambda x}\right)}{-\log(1-\phi)\{1-e^{-\theta}-\phi[1-\exp(-\theta e^{-\lambda x})]\}},\quad x>0, \tag{1.48}$$

where θ, λ, both positive and $0<\phi<1$ are parameters.

2. Characterization Results

As mentioned in the Introduction, characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) conditional expectation of a single function of the random variable. It should be mentioned that for the characterization (i) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1 Characterizations based on two truncated moments

In this subsection, we present characterizations of all the distributions mentioned in the Introduction, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to (Glänzel, 1987), see Theorem 2.1.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

Theorem 2.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty, e = \infty$ mightaswellbeallowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)} \right| \exp\big(-s(u)\big) du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Here is our first characterization.

Proposition 2.1.1. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[1 - \overline{\theta G}(x)^k\right]^{1+\frac{1}{k}}$ and $q_2(x) = q_1(x)\overline{G}(x)$ for $x > 0$. The random variable X belongs to (H-GC) family (1.2) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\overline{G}(x), \qquad x > 0. \tag{2.1.1}$$

Proof. Let X be a random variable with pdf (1.2), then

$$\big(1 - F(x)\big)E[q_1(x)|X \geq x] = \theta^{1/k}\overline{G}(x), \quad x > 0,$$

and

$$\big(1 - F(x)\big)E[q_2(x)|X \geq x] = \theta^{1/k}\overline{G}(x)^2, \quad x > 0,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{1}{2}q_1(x)\overline{G}(x) < 0 \quad for \;\; x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{g(x)}{\overline{G}(x)}, \quad x > 0,$$

and hence

$$s(x) = -\ln\left(\overline{G}(x)\right), \quad x > 0.$$

Now, in view of Theorem 2.1.1, X has density (1.2).

Corollary 2.1.1. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.1. The pdf of X is (1.2) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{g(x)}{\overline{G}(x)}, \quad x > 0. \tag{2.1.2}$$

The general solution of the differential equation (2.1.2) is

$$\eta(x) = \left[\overline{G}(x)\right]^{-1}\left[-\int g(x)\left(q_1(x)\right)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.2) is given in Proposition 2.1.1 with $D = \frac{1}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.1.1.

The proofs of the following Propositions (in this subsection) are similar to that of Proposition 2.1.1, so we only state the Propositions with their corresponding Corollaries. Further, we will not repeat the last sentence of the above paragraph for other distributions.

Proposition 2.1.2. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_2(x) = \left[1 - \overline{\theta}(1 + x^c)^{-kb}\right]^{1+\frac{1}{k}}$ and $q_1(x) = q_2(x)(1 + x^c)^{-1}$ for $x > 0$. The random variable X belongs to (HEBXII) family (1.4) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{b}{b+1}(1 + x^c), \qquad x > 0. \tag{2.1.3}$$

Corollary 2.1.2. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.2. The pdf of X is (1.4) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = -bcx^{c-1}(1 + x^c)^{-1}, \quad x > 0. \tag{2.1.4}$$

The general solution of the differential equation (2.1.4) is

$$\eta(x) = (1 + x^c)^{-b}\left[\int bcx^{c-1}(1 + x^c)^{b-1}\big(q_1(x)\big)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.4) is given in Proposition 2.1.2 with $D = 0$.

Proposition 2.1.3. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $q_2(x) = \left\{1 - \overline{\theta}\left[1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right]^k\right\}^{1+\frac{1}{k}}$ and $q_1(x) = q_2(x)\left(1 - e^{-\lambda x}\right)^{\alpha}$ for $x > 0$. The random variable X belongs to (HEEE) family (1.6) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left[1 + \left(1 - e^{-\lambda x}\right)^{\alpha}\right], \qquad x > 0. \tag{2.1.5}$$

Corollary 2.1.3. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.3. The pdf of X is (1.6) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}}{1 - (1 - e^{-\lambda x})^{\alpha}}, \quad x > 0. \tag{2.1.6}$$

The general solution of the differential equation (2.1.6) is

$$\eta(x) = \left[1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right]^{-1}\left[-\int \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1}\big(q_1(x)\big)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.6) is given in Proposition 2.1.3 with $D = \frac{1}{2}$.

Proposition 2.1.4. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[1 - \exp\left\{-\left[ax^b\left(e^{cx^d} - 1\right)\right]\right\}\right]^{1-\theta}$ and $q_2(x) = q_1(x)\exp\left\{-\left[ax^b\left(e^{cx^d} - 1\right)\right]\right\}$ for $x > 0$. The random variable X belongs to (H-GC) family (1.8) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\exp\left\{-\left[ax^b\left(e^{cx^d} - 1\right)\right]\right\}, \qquad x > 0. \tag{2.1.7}$$

Corollary 2.1.4. Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.4. The pdf of X is (1.8) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = ab\theta x^{b-1}e^{-ax^b\left(e^{cx^d}-1\right)+cx^d} \times \left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right), \quad x > 0. \tag{2.1.8}$$

The general solution of the differential equation (2.1.8) is

$$\eta(x)=e^{ax^b\left(e^{cx^d}-1\right)}\left[-\int ab\theta x^{b-1}e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.8) is given in Proposition 2.1.4 with $D=0$.

Proposition 2.1.5. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}\right\}^{1-b}\left[1-(1+\lambda x)^{-\theta}\right]^{-a\beta+\beta}$ and $q_2(x)=q_1(x)\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}$ for $x>0$. The random variable X belongs to (BEL) family (1.10) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}\left\{1+\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}\right\},\quad x>0, \tag{2.1.9}$$

Corollary 2.1.5. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.5. The pdf of X is (1.10) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\beta\theta\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\beta-1}}{1-[1-(1+\lambda x)^{-\theta}]^{\beta}},\qquad x>0. \tag{2.1.10}$$

The general solution of the differential equation (2.1.10) is

$$\eta(x)=\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}\right\}^{-1}\times\left[-\int\beta\theta\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\beta-1}\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.10) is given in Proposition 2.1.5 with $D=\frac{1}{2}$.

Proposition 2.1.6. Let $X:\Omega\to(a,b)$ be a continuous random variable and let $q_1(x)=1$ and $q_2(x)=\exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}$ for $a<x<b$. The random variable X belongs to (K) family (1.12) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}\exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\},\quad a<x<b. \tag{2.1.11}$$

Corollary 2.1.6. Let $X:\Omega\to(a,b)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.6. The pdf of X is (1.12) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\lambda\beta(b-a)(x-a)^{\beta-1}}{(b-x)^{\beta+1}},a<x<b. \tag{2.1.12}$$

The general solution of the differential equation (2.1.12) is

$$\eta(x)=\exp\left\{\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}\left[\begin{array}{l}-\int\frac{\lambda\beta(b-a)(x-a)^{\beta-1}}{(b-x)^{\beta+1}}\times\\ \exp\left\{-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}q_2(x)dx+D\end{array}\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.12) is given in Proposition 2.1.6 with $D=0$.

Proposition 2.1.7. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=\left\{1-(1-\beta)\exp\{1-(1+\lambda x)^{\alpha}\}\right\}^2$ and $q_2(x)=q_1(x)\exp\{-(1+\lambda x)^{\alpha}\}$ for $x>0$. The random variable X belongs to (TPEE) family (1.14) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}\exp\{-(1+\lambda x)^{\alpha}\},\quad x>0, \tag{2.1.13}$$

Corollary 2.1.7. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.7. The pdf of X is (1.14) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\alpha\lambda(1+\lambda x)^{\alpha},\quad x>0. \tag{2.1.14}$$

The general solution of the differential equation (2.1.14) is

$$\eta(x)=\exp\{(1+\lambda x)^{\alpha}\}\left[-\int\alpha\lambda(1+\lambda x)^{\alpha-1}\exp\{-(1+\lambda x)^{\alpha}\}\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.14) is given in Proposition 2.1.7 with $D=0$.

Proposition 2.1.8. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=\left(1-\overline{\alpha}e^{-\lambda x}-\alpha pe^{-\beta x}\right)^2$ and $q_2(x)=q_1(x)e^{-\beta x}$ for $x>0$. The random variable X belongs to (MEG) family (1.16) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}e^{-\beta x},\quad x>0, \tag{2.1.15}$$

Corollary 2.1.8. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.8. The pdf of X is (1.16) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\beta,\quad x>0. \tag{2.1.16}$$

The general solution of the differential equation (2.1.16) is

$$\eta(x) = e^{\beta x}\left[-\int \beta e^{-\beta x}\left(q_1(x)\right)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.16) is given in Proposition 2.1.8 with $D = 0$.

Proposition 2.1.9. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = \left\{1 - \left[1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\right]^a\right\}^{1-b}$ and $q_2(x) = q_1(x)e^{-e^{\alpha x - \frac{\beta}{x}}}$ for $x > 0$. The random variable X belongs to (KFWE) family (1.18) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}e^{-e^{\alpha x - \frac{\beta}{x}}}, \quad x > 0, \tag{2.1.17}$$

Corollary 2.1.9. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.9. The pdf of X is (1.18) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}} \quad x > 0. \tag{2.1.18}$$

The general solution of the differential equation (2.1.18) is

$$\eta(x) = e^{e^{\alpha x - \frac{\beta}{x}}}\left[-\int \left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}}e^{-e^{\alpha x - \frac{\beta}{x}}}\left(q_1(x)\right)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.18) is given in Proposition 2.1.9 with $D = 0$.

Proposition 2.1.10. Let $X:\Omega \to (\ln k,\infty)$ be a continuous random variable and let $q_1(x) = [1 - \lambda(1 - k^a e^{-ax})]^{-1}$ and $q_2(x) = q_1(x)e^{-ax}$ for $x > lnk$. The random variable X belongs to (TEP-I) family (1.20) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}e^{-ax}, \quad x > lnk \tag{2.1.19}$$

Corollary 2.1.10. Let $X:\Omega \to (\ln k,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.10. The pdf of X is (1.20) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = a, \quad x > lnk. \tag{2.1.20}$$

The general solution of the differential equation (2.1.20) is

$$\eta(x) = e^{ax}\left[-\int ae^{-ax}\left(q_1(x)\right)^{-1} q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.20) is given in Proposition 2.1.10 with $D = 0$.

Proposition 2.1.11. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = \left[1-\lambda+2\lambda e^{-\theta(e^{\alpha x}-1)}\right]^{-1}$ and $q_2(x) = q_1(x)e^{-\theta(e^{\alpha x}-1)}$ for $x>0$. The random variable X belongs to (TG) family (1.22) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}e^{-\theta(e^{\alpha x}-1)}, \quad x>0, \tag{2.1.21}$$

Corollary 2.1.11. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.11. The pdf of X is (1.22) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)} = \theta e^{\alpha x}, \quad x>0. \tag{2.1.22}$$

The general solution of the differential equation $(2.1.22)$ is

$$\eta(x) = e^{\theta(e^{\alpha x}-1)}\left[-\int \theta e^{\alpha x}e^{-\theta(e^{\alpha x}-1)}\left(q_1(x)\right)^{-1}q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.22)$ is given in Proposition 2.1.11 with $D=0$.

Proposition 2.1.12. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = \left[1+(1+x^c)^\lambda\right]^{-1}$ and $q_2(x)=1$ for $x>0$. The random variable X belongs to (EBXII) family (1.24) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = 2\left\{1+(1+x^c)^\lambda\right\}, \quad x>0, \tag{2.1.23}$$

Corollary 2.1.12. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.12. The pdf of X is (1.24) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)} = \frac{2\lambda c x^{c-1}(1+x^c)^{\lambda-1}}{1+(1+x^c)^\lambda}, \quad x>0. \tag{2.1.24}$$

The general solution of the differential equation $(2.1.24)$ is

$$\eta(x) = \left\{1+(1+x^c)^\lambda\right\}^2\left[-\int \frac{2\lambda c x^{c-1}(1+x^c)^{\lambda-1}}{\{1+(1+x^c)^\lambda\}^3}\left(q_1(x)\right)^{-1}q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.24)$ is given in Proposition 2.1.12 with $D=0$.

Proposition 2.1.13. Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x) = 1$ and $q_2(x) = \exp\left\{-\left(e^{(\alpha/x)^\beta}-1\right)^{-b}\right\}$ for $x>0$. The random variable X belongs to (WFr) family (1.26) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{a}{a+1}\exp\left\{-\left(e^{(\alpha/x)^\beta}-1\right)^{-b}\right\}, \quad x>0, \tag{2.1.25}$$

Corollary 2.1.13. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.13. The pdf of X is (1.26) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{ab\beta\alpha^{\beta}x^{-\beta-1}e^{(\alpha/x)^{\beta}}}{\left(e^{(\alpha/x)^{\beta}} - 1\right)^{b+1}}, \quad x > 0. \tag{2.1.26}$$

The general solution of the differential equation $(2.1.26)$ is

$$\eta(x) = \exp\left\{a\left(e^{(\alpha/x)^{\beta}} - 1\right)^{-b}\right\}\begin{bmatrix} -\int \frac{ab\beta\alpha^{\beta}x^{-\beta-1}e^{(\alpha/x)^{\beta}}}{\left(e^{(\alpha/x)^{\beta}} - 1\right)^{b+1}} \times \\ \exp\left\{-a\left(e^{(\alpha/x)^{\beta}} - 1\right)^{-b}\right\}\left(q_1(x)\right)^{-1}q_2(x)dx + D\end{bmatrix},$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.26)$ is given in Proposition 2.1.13 with $D = 0$.

Proposition 2.1.14. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[1 - (1-\alpha)Q\left(1 + \frac{\xi}{k}, \frac{x^k}{\lambda^k}\right)\right]^2 x^{-\xi}$ and $q_2(x) = q_1(x)e^{-\lambda^{-k}x^k}$ for $x > 0$. The random variable X belongs to (MOP $_g$W $_\alpha$) family (1.28) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}e^{-\lambda^{-k}x^k}, \quad x > 0, \tag{2.1.27}$$

Corollary 2.1.14. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.14. The pdf of X is (1.28) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = k\lambda^{-k}x^{k-1}, \quad x > 0. \tag{2.1.28}$$

The general solution of the differential equation $(2.1.28)$ is

$$\eta(x) = e^{\lambda^{-k}x^k}\left[-\int k\lambda^{-k}x^{k-1}e^{-\lambda^{-k}x^k}\left(q_1(x)\right)^{-1}q_2(x)dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.28)$ is given in Proposition 2.1.14 with $D = 0$.

Proposition 2.1.15. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 + \lambda - 2\lambda\left[\frac{(1-p)\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}}\right]\right\}\left[1 - p\left(1 - e^{-(\alpha x)^{\beta}}\right)^{\theta}\right]^2$ and $q_2(x) = q_1(x)\left(1 - e^{-(\alpha x)^{\beta}}\right)^{\theta}$ for $x > 0$. The random variable X belongs to (TEWG) family (1.30) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left(1 - e^{-(\alpha x)^{\beta}}\right)^{\theta}\right\}, \quad x > 0, \tag{2.1.29}$$

Corollary 2.1.15. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.15. The pdf of X is (1.30) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = -\frac{\beta\theta\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}}}{1 - e^{-(\alpha x)^{\beta}}}, \quad x > 0. \tag{2.1.30}$$

The general solution of the differential equation $(2.1.30)$ is

$$\eta(x) = \left(1 - e^{-(\alpha x)^{\beta}}\right)^{-1} \left[\int \beta\theta\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}} \left(q_1(x)\right)^{-1} q_2(x) dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.30)$ is given in Proposition 2.1.15 with $D = 0$.

Proposition 2.1.16. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 + \lambda - 2\lambda \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta}\right\}^{-1}$ and $q_2(x) = q_1(x) \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta}$ for $x > 0$. The random variable X belongs to (TGG) family (1.32) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta}\right\}, \quad x > 0, \tag{2.1.31}$$

Corollary 2.1.16. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.16. The pdf of X is (1.32) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\alpha\beta e^{\xi x}\left(\exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right)\left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta-1}}{1 - \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta}}, x > 0. \tag{2.1.32}$$

The general solution of the differential equation $(2.1.32)$ is

$$\eta(x) = \left\{1 - \left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta}\right\}^{-1} \times \left[-\int \alpha\beta e^{\xi x} \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\left[1 - \exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x} - 1\right)\right\}\right]^{\beta-1} \left(q_1(x)\right)^{-1} q_2(x) dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.32)$ is given in Proposition 2.1.16 with $D = \frac{1}{2}$.

Proposition 2.1.17. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [1 + \theta\xi F_{BS}(x)]^{\frac{1}{\xi}+1}$ and $q_2(x) = q_1(x) F_{BS}(x)$ for $x > 0$. The random variable X belongs to (NBBS) family (1.34) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left(1 + F_{BS}(x)\right), \quad x > 0, \tag{2.1.33}$$

Corollary 2.1.17. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.17. The pdf of X is (1.34) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{f_{BS}(x)}{1 - F_{BS}(x)}, \quad x > 0. \tag{2.1.34}$$

The general solution of the differential equation (2.1.34) is

$$\eta(x) = \left(1 - F_{BS}(x)\right)^{-1} \left[-\int f_{BS}(x) \left(q_1(x)\right)^{-1} q_2(x) dx + D \right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.34) is given in Proposition 2.1.17 with $D = \frac{1}{2}$.

Proposition 2.1.18. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = [1 - \overline{\alpha\gamma}(\lambda + 1, \theta x^2)]^2 x^{-2\lambda}$ and $q_2(x) = q_1(x) e^{-\theta x^2}$ for $x > 0$. The random variable X belongs to (NBBS) family (1.36) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2} e^{-\theta x^2}, \quad x > 0, \tag{2.1.35}$$

Corollary 2.1.18. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.18. The pdf of X is (1.36) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = 2\theta x, \quad x > 0. \tag{2.1.36}$$

The general solution of the differential equation (2.1.36) is

$$\eta(x) = e^{\theta x^2} \left[-\int 2\theta x e^{-\theta x^2} \left(q_1(x)\right)^{-1} q_2(x) dx + D \right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.36) is given in Proposition 2.1.18 with $D = 0$.

Proposition 2.1.19. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = x^{\alpha}(1 + x^{\alpha})^{-1}$ and $q_2(x) = q_1(x) e^{-\theta x^{-\alpha}}$ for $x > 0$. The random variable X belongs to (GIL) family (1.38) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left[1 + e^{-\theta x^{-\alpha}}\right], \quad x > 0, \tag{2.1.37}$$

Corollary 2.1.19. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.19. The pdf of X is (1.38) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha\theta x^{-\alpha-1}e^{-\theta x^{-\alpha}}}{1-e^{-\theta x^{-\alpha}}},\quad x>0. \tag{2.1.38}$$

The general solution of the differential equation $(2.1.38)$ is

$$\eta(x)=\left(1-e^{-\theta x^{-\alpha}}\right)^{-1}\left[-\int \alpha\theta x^{-\alpha-1}e^{-\theta x^{-\alpha}}\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.38)$ is given in Proposition 2.1.19 with $D=0$.

Proposition 2.1.20. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)=\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta(1-a)}\left\{\begin{matrix}1-\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{a\delta}\times\\ \left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\end{matrix}\right\}^{b-1}\left\{1+\lambda-2\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right\}^{-1}$ and $q_2(x)=q_1(x)\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}$ for $x>0$. The random variable X belongs to (Kw-TEAW) family (1.40) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}\left\{1+\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\right\},\qquad x>0, \tag{2.1.39}$$

Corollary 2.1.20. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.20. The pdf of X is (1.40) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}= -\frac{a\delta\lambda e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\times\left[1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right]^{\delta-1}\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a-1}}{1-\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}},x>0. \tag{2.1.40}$$

The general solution of the differential equation $(2.1.40)$ is

$$\eta(x)=\left\{1-\left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\right\}^{-1}\times$$
$$\left[\begin{matrix}\int a\delta\lambda e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\left[1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right]^{\delta-1}\times\\ \left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a-1}\left(q_1(x)\right)^{-1}q_2(x)dx+D\end{matrix}\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.40)$ is given in Proposition 2.1.20 with $D=\frac{1}{2}$.

Proposition 2.1.21. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a(1-\theta)} \left\{1 - \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{\theta}\right\}^{1-b}$ and $q_2(x) = q_1(x)\left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a}$ for $x > 0$. The random variable X belongs to (BEG $_1$) family (1.42) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a}\right\}, \quad x > 0, \tag{2.1.41}$$

Corollary 2.1.21. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.21. The pdf of X is (1.42) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{a\lambda^2 x e^{-\lambda x}\left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a-1}}{1 - [1 - e^{-\lambda x}(1 + \lambda x)]^{a}}, x > 0. \tag{2.1.42}$$

The general solution of the differential equation (2.1.42) is

$$\eta(x) = \left\{1 - \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a}\right\}^{-1} \times \left[-\int a\lambda^2 x e^{-\lambda x}\left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a-1}\left(q_1(x)\right)^{-1} q_2(x) dx + D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.42) is given in Proposition 2.1.21 with $D = \frac{1}{2}$.

Proposition 2.1.22. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 - \left[1 - \left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]\right\}^{1-b}$ and $q_2(x) = q_1(x)\left[1 - \left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}$ for $x > 0$. The random variable X belongs to (KwKwW) family (1.44) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x) = \frac{1}{2}\left\{1 + \left[1 - \left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}, \quad x > 0, \tag{2.1.43}$$

Corollary 2.1.22. Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.22. The pdf of X is (1.44) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{ab\alpha\beta\theta\lambda^{\theta} x^{\theta-1} e^{-(\lambda x)^{\theta}}\left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha-1}\left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta-1} \times \left[1 - \left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a-1}}{1 - \left[1 - \left(1 - \left(1 - e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}}, \tag{2.1.44}$$

The general solution of the differential equation $(2.1.44)$ is

$$\eta(x)=\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{-1}\times$$
$$\left[\begin{array}{l}\int ab\alpha\beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta-1}\times\\ \left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a-1}\left(q_1(x)\right)^{-1}q_2(x)dx+D\end{array}\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.44)$ is given in Proposition 2.1.22 with $D=\frac{1}{2}$.

Without loss of generality, we assume $\mu=0$ and $\sigma=1$ in the following Proposition.

Proposition 2.1.23. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\{1-\exp[-e^{-x}]\}^{-\alpha(b-1)}\{1-\{1-\exp[-e^{-x}]\}^{\alpha}\}^{1-a}$ and $q_2(x)=q_1(x)\{1-\exp[-e^{-x}]\}$ for $x>0$. The random variable X belongs to (BEG_2) family (1.46) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(\mathrm{x})=\frac{a}{a+1}\{1-\exp[-e^{-x}]\},\quad x>0. \tag{2.1.45}$$

Corollary 2.1.23. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.23. The pdf of X is (1.46) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{ae^{-x}\exp[-e^{-x}]}{1-\exp[-e^{-x}]},\quad x>0. \tag{2.1.46}$$

The general solution of the differential equation $(2.1.46)$ is

$$\eta(x)=\{1-\exp[-e^{-x}]\}^{-1}\left[-\int ae^{-x}\exp[-e^{-x}]\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation $(2.1.46)$ is given in Proposition 2.1.23 with $D=0$.

Proposition 2.1.24. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\left\{1-e^{-\theta}-\phi\left[1-\exp\left(-\theta e^{-\lambda x}\right)\right]\right\}$ and $q_2(x)=q_1(x)\exp\left(-\theta e^{-\lambda x}\right)$ for $x>0$. The random variable X belongs to (EPL) family (1.48) if and only if the function η defined in Theorem 2.1.1 has the form

$$\eta(x)=\frac{1}{2}\left\{1+\exp\left(-\theta e^{-\lambda x}\right)\right\},\quad x>0, \tag{2.1.47}$$

Corollary 2.1.24. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1.24. The pdf of X is (1.48) if and only if there exist functions q_2 and η defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\theta\lambda\exp\left(-\lambda x-\theta e^{-\lambda x}\right)}{\{1-\exp(-\theta e^{-\lambda x})\}},\quad x>0. \tag{2.1.48}$$

The general solution of the differential equation (2.1.48) is

$$\eta(x)=\left\{1-\exp\left(-\theta e^{-\lambda x}\right)\right\}^{-1}\left[-\int \theta\lambda\exp\left(-\lambda x-\theta e^{-\lambda x}\right)\left(q_1(x)\right)^{-1}q_2(x)dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (2.1.48) is given in Proposition 2.1.24 with $D=\frac{1}{2}$.

2.2 Characterization based on hazard function

It is well known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x). \tag{2.2.1}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following Propositions establish non-trivial characterizations of (H-GC), (HEBXII), (HEEE), (EGWG; for $\theta=1$), (W-GF), (K), (TPEE), (MEG), (KFWE; for $a=1$), (TEP-I), (TG), (EBXII), (WFr), (TGG), (MOEGR), (BEG $_1$) and (BEG $_2$) distributions in terms of the hazard function which are not of the trivial form given in (2.2.1).

Proposition 2.2.1. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable. The pdf of X is (1.2) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x)-\frac{g'(x)}{g(x)}h_F(x)=-\frac{g(x)\left\{(k+1)\overline{\theta G}(x)^k-1\right\}}{\overline{G}(x)^2\left[1-\overline{\theta G}(x)^k\right]^2},\qquad x>0. \tag{2.2.2}$$

with the boundary condition $h_F(0)=\theta^{-1}\lim_{x\rightarrow0^+}g(x)$.

Proof. If X has pdf (1.2), then clearly (2.2.2) holds. Now, if (2.2.2) holds, then

$$\frac{d}{dx}\left\{\left(g(x)\right)^{-1}h_F(x)\right\}=\frac{d}{dx}\left\{\left\{\overline{G}(x)\left[1-\overline{\theta G}(x)^k\right]\right\}^{-1}\right\},$$

or, equivalently,

$$h_F(x)=\frac{g(x)}{\overline{G}(x)\left[1-\overline{\theta G}(x)^k\right]},$$

which is the hazard function of the (H-GC) distribution.

The proofs of the following Propositions in this subsection are similar to that of Proposition 2.2.1, so we state them without proofs.

Proposition 2.2.2. Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable. The pdf of X is (1.4) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x)-(c-1)x^{-1}h_F(x)$$
$$=\frac{bc^2x^{2(c-1)}(1+x^c)^{-2}\left[\overline{\theta}(1-kb)(1+x^c)^{-kb}-1\right]}{\left[1-\overline{\theta}(1+x^c)^{-kb}\right]^2}$$

$$= bc\frac{d}{dx}\left\{\frac{(1+x^c)^{-1}}{1-\overline{\theta}(1+x^c)^{-kb}}\right\}, \quad x > 0. \tag{2.2.3}$$

with the boundary condition $h_F(0) = 0$ for $c > 1$.

Proposition 2.2.3. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) + \lambda h_F(x)$$
$$= \alpha\lambda e^{-\lambda x}\frac{d}{dx}\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha-1}}{[1-(1-e^{-\lambda x})^{\alpha}] - \overline{\theta}[1-(1-e^{-\lambda x})^{\alpha}]^{k+1}}\right\}, \tag{2.2.4}$$

$x > 0$, with the boundary condition $h_F(0) = 0$ for $\alpha > 1$.

Proposition 2.2.4. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.8) for $\theta = 1$,if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - (b-1)x^{-1}h_F(x) = ax^{b-1}\left\{cdx^{d-1}(b+d+cdx^d)e^{cx^d}\right\}, \tag{2.2.5}$$

$x > 0$, with the boundary condition $h_F(0) = 0$ for $b > 1$.

Proposition 2.2.5. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. For $a = 1$, the pdf of X is (1.10), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) + (\theta+1)(1+\lambda x)^{-1}h_F(x)$$
$$= \frac{\theta\lambda(\beta-1)(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\beta-2}}{\{1-[1-(1+\lambda x)^{-\theta}]^{\beta}\}^2}, x > 0. \tag{2.2.6}$$

Proposition 2.2.6. Let $X: \Omega \to (a, b)$ be a continuous random variable. The pdf of X is (1.12), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - \frac{(\beta-1)}{(x-a)}h_F(x) = \frac{\lambda\beta(\beta+1)(b-a)(x-a)^{\beta-1}}{(b-x)^{\beta+2}}, a < x < b, \tag{2.2.7}$$

with initial condition $h_F(a) = 0$ for $\beta > 1$.

Proposition 2.2.7. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.14), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - \lambda(\alpha-1)(1+\lambda x)^{-1}h_F(x)$$
$$= -\frac{\alpha^2\lambda^2(1-\beta)(1+\lambda x)^{2(\alpha-1)}\exp\{1-(1+\lambda x)^{\alpha}\}}{\left\{1-(1-\beta)\exp\{1-(1+\lambda x)^{\alpha}\}\right\}^2}$$
$$= \frac{d}{dx}\left\{\frac{\alpha\lambda}{1-(1-\beta)\exp\{1-(1+\lambda x)^{\alpha}\}}\right\}, \quad x > 0, \tag{2.2.8}$$

with initial condition $h_F(0) = \frac{\alpha\lambda}{\beta}$.

Proposition 2.2.8. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.16), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \beta h_F(x) = -\beta^2 \left(1 - \overline{\alpha} e^{-\lambda x} - \alpha p e^{-\beta x}\right)^{-2}, \quad x > 0, \tag{2.2.9}$$

with initial condition $h_F(0) = \frac{\beta}{\alpha(1-p)}$.

Proposition 2.2.9. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.18), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \left(\alpha + \frac{\beta}{x^2}\right) h_F(x) = -\frac{2\beta b}{x^3} e^{\alpha x - \frac{\beta}{x}}, \quad x > 0, \tag{2.2.10}$$

with initial condition $h_F(1) = b(\alpha + \beta) e^{\alpha - \beta}$.

Proposition 2.2.10. Let $X: \Omega \to (\ln k, \infty)$ be a continuous random variable. The pdf of X is (1.20), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) + a h_F(x) = a^2 + \frac{a^2 \lambda^2 k^{2a} e^{-2ax}}{[1 - \lambda(1 - k^a e^{-ax})]^2}, \quad x > ln k, \tag{2.2.11}$$

with initial condition $h_F(\ln k) = \alpha(1 + \lambda)$.

Proposition 2.2.11. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.22), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \alpha h_F(x) = -\frac{\theta^2 \lambda (1 - \lambda) e^{2\alpha x} e^{\theta(e^{\alpha x} - 1)}}{[\lambda + (1 - \lambda) e^{\theta(e^{\alpha x} - 1)}]^2}, \quad x > 0, \tag{2.2.12}$$

with initial condition $h_F(0) = \theta(1 + \lambda)$.

Proposition 2.2.12. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.24), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - (c - 1) x^{-1} h_F(x) = \frac{\lambda c^2 x^{2(c-1)} (1 + x^c)^{\lambda - 2} \left\{\lambda - 1 - (1 + x^c)^\lambda\right\}}{[1 + (1 + x^c)^\lambda]^2}, \quad x > 0, \tag{2.2.13}$$

with initial condition $h_F(0) = 0$ for $c > 1$.

Proposition 2.2.13. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (1.26), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) + (\beta + 1) x^{-1} h_F(x) = \frac{\beta \alpha^\beta x^{-\beta - 1} e^{-b(\alpha/x)^\beta} \left[b + e^{-(\alpha/x)^\beta}\right]}{\left[1 - e^{-(\alpha/x)^\beta}\right]^{b+2}}, \quad x > 0, \tag{2.2.14}$$

with initial condition $h_F(\alpha) = \frac{ab\beta e}{\alpha(e-1)^{b+1}}$.

Proposition 2.2.14. Let $X: \Omega \to (0, \infty)$ be a continuous random variable. For $\beta = 1$, the pdf of X is (1.32), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \xi h_F(x)$$

$$= \alpha\lambda e^{\xi x}\frac{d}{dx}\left\{\frac{\exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x}-1\right)\right\}}{1-\lambda+\lambda\exp\left\{-\frac{\alpha}{\xi}\left(e^{\xi x}-1\right)\right\}}\right\}, \quad x>0, \tag{2.2.15}$$

with initial condition $h_F(0) = \alpha(1+\lambda)$.

Proposition 2.2.15. Let $X: \Omega \to (0,\infty)$ be a continuous random variable. The pdf of X is (1.36), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) + x^{-1}h_F(x)$$
$$= \frac{2\theta^{\lambda+1}x^{2\lambda}}{\Gamma(\lambda+1)}\frac{d}{dx}\left\{\frac{e^{-\theta x}}{\overline{\gamma}(\lambda+1,\theta x^2)[1-\overline{\alpha\gamma}(\lambda+1,\theta x^2)]}\right\}, x>0, \tag{2.2.16}$$

with initial condition $h_F(1) = \frac{\theta^{\lambda+1}e^{-\theta}}{\Gamma(\lambda+1)\overline{\gamma}(\lambda+1,\theta)[1-\overline{\gamma}(\lambda+1,\theta x)]}$.

Proposition 2.2.16. Let $X: \Omega \to (0,\infty)$ be a continuous random variable. For $b=1$ and $a<2$, the pdf of X is (1.42), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - x^{-1}h_F(x)$$
$$= \frac{\theta\lambda^2 x}{(2-a)B(a,b)}\frac{d}{dx}\left\{e^{-\lambda x}\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta(a-1)-1}\right\}, x>0. \tag{2.2.17}$$

Proposition 2.2.17. Let $X: \Omega \to (0,\infty)$ be a continuous random variable. For $a=1$, the pdf of X is (1.46), if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) + h_F(x) = \frac{\alpha b e^{-2x}\exp[-e^{-x}]}{\{1-\exp[-e^{-x}]\}^2}, \quad x>0. \tag{2.2.18}$$

2.3 Characterization based on conditional expectation

The following propositions have already appeared in (Hamedani, 2013) , so we will just state them here which can be used to characterize (H-GC), (HEBXII), (HEEE), (EGWG), (K), (MEG), (KFWE), (WFr) and (Kw-TEAW) distributions.

Proposition 2.3.1. Let $X: \Omega \to (a,b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+}\psi(x) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X)|X \geq x] = \delta\psi(x), \quad x \in (a,b),$$

if and only if

$$\psi(x) = \left(1-F(x)\right)^{\frac{1}{\delta}-1}, \quad x \in (a,b).$$

Proposition 2.3.2. Let $X: \Omega \to (a,b)$ be a continuous random variable and let $\psi \in C^1(a,b)$ and $\varphi \in C^2(a,b)$ such that $\lim_{x\to a^-}\left(\int_a^x \frac{\varphi'(u)}{\varphi(u)-\psi(u)}du\right) = \infty$. Then

$$E[\psi(X)|X \geq x] = \varphi(x).$$

implies

$$1-F(x) = \exp\left\{-\int_a^x \frac{\varphi'(u)}{\varphi(u)-\psi(u)}du\right\}.$$

Remarks 2.3.1. (a) It is easy to see that for certain functions, e.g., $\psi(x)=\frac{\theta\overline{G}(x)^k}{1-\overline{\theta}\overline{G}(x)^k}$, $\delta=\frac{1}{k+1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (H-GC) distribution. (b) Taking, e.g., $\psi(x)=\frac{\theta(1+x^c)^{-kb}}{1-\overline{\theta}(1+x^c)^{-kb}}$, $\delta=\frac{1}{k+1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (HEBXII) distribution. (c) Taking, e.g., $\psi(x)=\frac{\theta\left[1-\left(1-e^{-\lambda x}\right)^{\alpha}\right]^k}{\left\{1-\overline{\theta}\left[1-\left(1-e^{-\lambda x}\right)^{\alpha}\right]^k\right\}}$, $\delta=\frac{1}{k+1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (HEEE) distribution. (d) Taking, e.g., $\psi(x)=\exp\left\{\left[ax^b\left(e^{cx^d}-1\right)\right]\right\}$, $\delta=\frac{1}{2}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (EGWG) distribution for $\theta=1$. (e) Taking, e.g., $\psi(x)=\exp\left\{\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}$, $\delta=\frac{\alpha}{\alpha-1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (W-GF) distribution for $\alpha\neq1$. (f) Taking, e.g., $\psi(x)=\exp\left\{-\left(\frac{x-a}{b-x}\right)^{\beta}\right\}$, $\delta=\frac{\lambda}{1+\lambda}$ and $(a,b)=(a,b)$, Proposition 2.3.1 provides a characterization of (K) distribution. (g) Taking, e.g., $\psi(x)=\frac{[\alpha(1-p)]^{1/\beta}e^{-x}}{\left(1-\overline{\alpha}e^{-\lambda x}-\alpha pe^{-\beta x}\right)^{1/\beta}}$, $\delta=\frac{\beta}{1+\beta}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (MEG) distribution. (h) Taking, e.g., $\psi(x)=1-\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{a}$, $\delta=\frac{b}{1+b}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (KFWE) distribution. (i) Taking, e.g., $\psi(x)=\exp\left\{-\left(e^{(\alpha/x)^{\beta}}-1\right)^{-b}\right\}$, $\delta=\frac{a}{1+a}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (WFr) distribution. (j) Taking e.g., $\psi(x)=\left\{\begin{array}{c}1-\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{a\delta}\times\\ \left[1+\lambda-\lambda\left(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})}\right)^{\delta}\right]^{a}\end{array}\right\}$, $\delta=\frac{b}{b+1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (Kw-TEAW) distribution. (k) Taking, e.g., $\psi(x)=1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]$, $\delta=\frac{b}{b+1}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (KwKwW) distribution. (l) Taking, e.g., $\psi(x)=1-\exp[-e^{-x}]$, $\delta=\frac{\alpha b}{1+\alpha b}$ and $(a,b)=(0,\infty)$, Proposition 2.3.1 provides a characterization of (EPL) distribution. (m) Our choices of the functions in $(a)-(l)$ are clearly for the sake of simplicity.

Remarks 2.3.2. (k) Taking, e.g., $\frac{1}{2}(\psi(x))=\varphi(x)=\exp\left\{ax^b\left(e^{cx^{\lambda}}-1\right)\right\}$, Proposition 2.3.2 provides a characterization of (EGWG) distribution for $\theta=1$. (l) Taking, e.g., $\frac{1}{2}(\psi(x))=\varphi(x)=\exp\left\{\alpha\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right\}$, Proposition 2.3.2 provides a characterization of (W-GF) distribution. (m) Taking, e.g., $\frac{1}{2}(\psi(x))=\varphi(x)=\exp\left\{\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right\}$, Proposition 2.3.2 provides a characterization of (K) distribution. (n)

Taking, e.g., $\frac{1}{2}(\psi(x)) = \varphi(x) = e^{be^{\alpha x} - \beta x^{-1}}$, Proposition 2.3.2 provides a characterization of (KFWE; for $a = 1$) distribution. (o) Taking, e.g., $\frac{1}{2}(\psi(x)) = \varphi(x) = \exp\left\{-\left(e^{(\alpha/x)^{\beta}} - 1\right)^{-b}\right\}$, Proposition 2.3.2 provides a characterization of (WFr) distribution. (p) Again, clearly, there are other suitable functions as well, we chose the above ones for the sake of simplicity.

3. Concluding Remarks

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with the characterizations of $tewnty\ four$ new univariate continuous distributions which have appeared in the literature in 2015-2016. We certainly hope that the content of this work will be useful to the investigators who are interested to know if they have chosen the right distributions.

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