A Multivariate Weibull Distribution

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Abstract

A multivariate survival function of Weibull Distribution is developed by expanding the theorem by Lu and Bhattacharyya. From the survival function, the probability density function, the cumulative probability function, the determinant of the Jacobian Matrix, and the general moment are derived.

Keywords: general moment; multivariate survival function; set partition; Weibull distribution; Copula; competing risks.

1. Introduction

Lu and Bhattacharyya (1990) developed a joint survival function by letting $h_1(x)$ and $h_2(y)$ be two arbitrary failure rate functions on $[0,\infty)$, and $H_1(x)$ and $H_2(y)$ be their corresponding cumulative failure rate. Given the stress *S*=*s* > 0, the joint survival function conditioned on *s*, as they defined, is

$$\overline{F}(x,y\,|\,s)=\exp\left\{-\left[H_1(x)+H_2(y)\right]^{\gamma}s\right\},$$

where γ measures the conditional association of *X* and *Y*. Further, based on the joint survival function, they proved a theorem that a bivariate survival function $\overline{F}(x,y\,|\,s)$ can be derived with the marginals $\overline{F}_x$ and $\overline{F}_y$ given the assumption that the Laplace transform of the stress *S* exists on $[0,\infty)$ and is strictly decreasing.

From the theorem, they derived a bivariate Weibull Distribution

$$\overline{F}(x,y)=\exp\left\{-\left[\left(\frac{x}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{y}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha}\right\}$$

where $0<\alpha\le 1$, $0<\lambda_1, \lambda_2<\infty$, and $0<\gamma_1, \gamma_2<\infty$. This bivariate Weibull Distribution is exactly the same as developed by Hougaard (1986).

Following the same steps, the theorem can be expanded to more than two random variables, and, therefore, a multivariate survival function of Weibull distribution is constructed as

$$S(x_1,x_2,...,x_n)=\exp\left\{-\left[\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+...+\left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}\right]^{\alpha}\right\} \tag{1}$$

where α measures the association among the variables, $0 < \alpha \le 1$, $0 < \lambda_1, \lambda_2, ..., \lambda_n < \infty$, and $0 < \gamma_1, \gamma_2, ..., \gamma_n < \infty$.

This model can also be derived by using a copula construction with the generator $\left(-\log(.)\right)^{1/\alpha}$ (Frees and Valdez 1998). Equation (1) is similar to the "genuine multivariate Weibull distribution" developed by Crowder (1989) who, in his paper, studied another version extended from the genuine multivariate Weibull distribution.

In this paper, we mathematically intensively studied the proposed multivariate Weibull model of Equation (1) by providing the probability density function, and the Jcaobian matrix in section 2, the general moment in section 3, an application in section 4 and conclusions in section 5.

2. Probability Density Function of The Multivariate Weibull Distribution

The multivariate probability density function $f(x_1,x_2,...,x_n)$ of a multivariate distribution function can be obtained by differentiating the multivariate survival function with respect to each variable. Li (1997), and Yi and Weng (2006) had shown that:

$$f(x_1,x_2,...,x_n)=(-1)^n\frac{\partial^n S(x_1,x_2,...,x_n)}{\partial x_1\partial x_2...\partial x_n}.$$

Using Li's derivation and one of the special cases of the multivariate Faa di Bruno formula by Constantine and Savits (1996), the probability density function is

$$\begin{aligned} f(x_1,x_2,...,x_n)=&\left(\frac{-1}{\alpha}\right)^n\exp\left\{-\left[\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+...+\left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}\right]^{\alpha}\right\}\\ &\cdot\left[\left(\frac{\gamma_1}{\lambda_1}\right)\left(\frac{\gamma_2}{\lambda_2}\right)...\left(\frac{\gamma_n}{\lambda_n}\right)\right]\left[\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}-1}\left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}-1}...\left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}-1}\right]\\ &\cdot\sum_{i=1}^{P(n)}\left\{(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\left[\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+...+\left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}\right]^{k_i\alpha-n}\right\} \end{aligned} \tag{2}$$

where k_i is the number of summands of the *i*th partition of *n* such that $n_1 + n_2 + \cdots + n_{k_i} = n$, $n_1 \ge n_2 \ge \cdots \ge n_{k_i} > 0$, $1 \le k_i \le n$; $\alpha^{\underline{n_j}}$ is equal to $\alpha(\alpha-1)...(\alpha-n_j+1)$, the falling factorial of α (Kunth 1992); $P(n)$ is the total number of partitions of *n*; $P_s(n,i)$ is the total number of set partitions of the set S_n={1,…,*n*} corresponding to the *i*th partition of *n*. The specific way of partitioning *n* and S_n is given by McCullagh and Wilks (1988). In their paper, partitions of *n* are in increasing number of summands and ordering all the summands in inverse lexicographic order when a partition has the same number of summands, and S_n= {1,…, *n*} is partitioned by "listing the blocks from the largest to the smallest and by breaking the ties of equal sized blocks by ordering them lexicographically" and the number of blocks in a set partition is equal to the number of summands of the corresponding partition of *n*. For example, the total number of blocks of the partition of S_n corresponding to the *i*th partition is k_i and the numbers of elements in each block are equal to $n_1, n_2, \cdots, n_{k_i}$.

Similar to the derivation of the Bivariate Weibull Distribution by Lu and Bhattacharyya (1990), let $(y_1, y_2, ..., y_n)$

$$=\left(\frac{z_1}{z_1+z_2+...+z_n}, \frac{z_2}{z_1+z_2+...+z_n}, ..., \frac{z_{n-1}}{z_1+z_2+...+z_n}, (z_1+z_2+...+z_n)^{\alpha}\right) \quad (3)$$

where $z_1 = \left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}, z_2 = \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}, \ldots, z_n = \left(\frac{x_n}{\lambda_n}\right)^{\frac{\gamma_n}{\alpha}}$.

Then, $x_1 = y_1^{\frac{\alpha}{\gamma_1}} y_n^{\frac{1}{\gamma_1}} \lambda_1, x_2 = y_2^{\frac{\alpha}{\gamma_2}} y_n^{\frac{1}{\gamma_2}} \lambda_2, \ldots, x_{n-1} = y_{n-1}^{\frac{\alpha}{\gamma_{n-1}}} y_n^{\frac{1}{\gamma_{n-1}}} \lambda_{n-1}$,

$$x_n = (1 - y_1 - y_2 - \cdots - y_{n-1})^{\frac{\alpha}{\gamma_n}} y_n^{\frac{1}{\gamma_n}} \lambda_n.$$

Note that $z_1, z_2, ..., z_n \ge 0$, and

$$1 - y_1 - y_2 - ... - y_{n-1}$$

$$=1 - \frac{z_1+z_2+...+z_{n-1}}{z_1+z_2+...+z_n}$$

$$=\frac{z_n}{z_1+z_2+...+z_n} \ge 0.$$

The Jacobian matrix is

$$J=\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$

Let $C(i,j)$ be the i^{th} row and j^{th} column in the Jacobian matrix, then

$C(i,i)=\dfrac{\alpha\lambda_i y_i^{\frac{\alpha}{\gamma_i}-1} y_n^{\frac{1}{\gamma_i}}}{\gamma_i}$, $i=1,2,\ldots,n-1$,

$C(i,n)=\dfrac{\lambda_i y_i^{\frac{\alpha}{\gamma_i}} y_n^{\frac{1}{\gamma_i}-1}}{\gamma_i}$, $i=1,2,\ldots,n-1$,

$C(n,i)=-\dfrac{\alpha\lambda_n\left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{\alpha}{\gamma_n}-1} y_n^{\frac{1}{\gamma_n}}}{\gamma_n}$, $i=1,2,\ldots,n-1$,

$C(n,n)=\dfrac{\lambda_n\left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{\alpha}{\gamma_n}} y_n^{\frac{1}{\gamma_n}-1}}{\gamma_n}$,

$C(i,j)=0$ when $i\neq j$, $i=2,\ldots,n-1$, $j=2,\ldots,n-1$.

The determinant of the Jacobian matrix can be obtained using Gaussian elimination to construct an upper triangle matrix. The determinant is then equal to the product of the diagonal elements and is given as

$$|J|=\left(\prod_{i=1}^{n-1}C(i,i)\right)\left(C(n,n)-\sum_{j=1}^{n-1}\frac{C(n,j)C(j,n)}{C(j,j)}\right)$$

$$=\frac{\alpha^{n-1}\lambda_1\lambda_2\cdots\lambda_n y_1^{\frac{\alpha}{\gamma_1}-1} y_2^{\frac{\alpha}{\gamma_2}-1}\cdots y_{n-1}^{\frac{\alpha}{\gamma_{n-1}}-1}\left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{\alpha}{\gamma_n}-1} y_n^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\cdots+\frac{1}{\gamma_n}-1}}{\gamma_1\gamma_2\cdots\gamma_n}. \tag{4}$$

After the derivation of the Jacobian, the PDF in terms of $y_1, y_2, \cdots, y_n$,

$$g\left(y_1,y_2,\cdots,y_n\right)=$$

$$f\left(y_1^{\frac{\alpha}{\gamma_1}} y_n^{\frac{1}{\gamma_1}}\lambda_1, y_2^{\frac{\alpha}{\gamma_2}} y_n^{\frac{1}{\gamma_2}}\lambda_2,\cdots, y_{n-1}^{\frac{\alpha}{\gamma_{n-1}}} y_n^{\frac{1}{\gamma_{n-1}}}\lambda_{n-1}, \left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{\alpha}{\gamma_n}} y_n^{\frac{1}{\gamma_n}}\lambda_n\right)|J|$$

$$=\left(-1\right)^{n}\left\{\sum_{i=1}^{P(n)}\left(\alpha^{-1}y_{n}^{\;k_i-1}\left(-1\right)^{k_i}P_{s}\left(n,i\right)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right)\right\}\exp(-y_{n})$$

$$=\left(\Gamma\left(n\right)y_{1}^{\;1-1}y_{2}^{\;1-1}...y_{n-1}^{\;\;1-1}\left(1-y_{1}-y_{2}-...-y_{n-1}\right)^{1-1}\right)f\left(y_{n}\right), \qquad (5)$$

where $y_1, y_2, \ldots, y_{n-1}$ has a Dirichlet distribution with the probability density equal to

$\Gamma\left(n\right)y_{1}^{\;1-1}y_{2}^{\;1-1}...y_{n-1}^{\;\;1-1}\left(1-y_{1}-y_{2}-...-y_{n-1}\right)^{1-1}$, and

$$f\left(y_{n}\right)=\frac{\left(-1\right)^{n}}{\Gamma\left(n\right)}\left\{\sum_{i=1}^{P(n)}\left(\alpha^{-1}y_{n}^{\;k_i-1}\left(-1\right)^{k_i}P_{s}\left(n,i\right)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right)\right\}\exp(-y_{n}) \qquad (6)$$

has a mixture distribution of the exponential distribution and Gamma distribution.

Equation (6) can be rewritten as:

$$f\left(y_{n}\right)=\frac{\left(-1\right)^{n}}{\Gamma\left(n\right)}\left\{\sum_{i=1}^{P(n)}\left(\alpha^{-1}y_{n}^{\;k_i-1}\left(-1\right)^{k_i}\Gamma\left(k_i\right)P_{s}\left(n,i\right)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right)\frac{\exp(-y_{n})}{\Gamma\left(k_i\right)}\right\}.$$

When it is integrated over the range of $y_{n,}$ it becomes:

$$\frac{\left(-1\right)^{n}}{\Gamma\left(n\right)}\left\{\sum_{i=1}^{P(n)}\left(\alpha^{-1}\left(-1\right)^{k_i}\Gamma\left(k_i\right)P_{s}\left(n,i\right)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right)\right\}=1.$$

That is the weights of y_n are summed to 1. The probability density function of y_n is the mixed Gamma distribution by Downton (1969). Following his derivation, the cumulative density function of y_n is:

$$1-\frac{\left(-1\right)^{n}}{\Gamma\left(n\right)}\left(\sum_{i=1}^{n}\frac{y_{n}^{\;i-1}}{\Gamma\left(i\right)}\sum_{k\geq i}^{P(n)}\left(\alpha^{-1}\left(-1\right)^{k}\Gamma\left(k\right)P_{s}\left(n,i\right)\left(\prod_{j=1}^{k}\alpha^{\underline{n_j}}\right)\right)\right)\exp(-y_{n}). \qquad (7)$$

3. The General Moment

The general moment of $x_1, x_2, \cdots, x_n$ is

$$E\left[x_{1}^{\;i_1}x_{2}^{\;i_2}\cdots x_{n}^{\;i_n}\right]=$$

$$E\left[\left(y_{1}^{\frac{\alpha}{\gamma_1}}y_{n}^{\frac{1}{\gamma_1}}\lambda_{1}\right)^{i_1}\left(y_{2}^{\frac{\alpha}{\gamma_2}}y_{n}^{\frac{1}{\gamma_2}}\lambda_{2}\right)^{i_2}\cdots\left(y_{n-1}^{\frac{\alpha}{\gamma_{n-1}}}y_{n}^{\frac{1}{\gamma_{n-1}}}\lambda_{n-1}\right)^{i_{n-1}}\left(\left(1-y_{1}-y_{2}-\cdots-y_{n-1}\right)^{\frac{\alpha}{\gamma_n}}y_{n}^{\frac{1}{\gamma_n}}\lambda_{n}\right)^{i_n}\right]$$

$$=\left(\lambda_{1}^{\;i_1}\lambda_{2}^{\;i_2}\cdots\lambda_{n}^{\;i_n}\right)E\left[y_{1}^{\frac{i_1\alpha}{\gamma_1}}y_{2}^{\frac{i_2\alpha}{\gamma_2}}\cdots y_{n-1}^{\frac{i_{n-1}\alpha}{\gamma_{n-1}}}\left(1-y_{1}-y_{2}-\cdots-y_{n-1}\right)^{\frac{i_n\alpha}{\gamma_n}}\right]E\left[y_{n}^{\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots\frac{i_n}{\gamma_n}}\right].$$

Considering the second factor,

$$E\left[y_1^{\frac{i_1\alpha}{\gamma_1}} y_2^{\frac{i_2\alpha}{\gamma_2}} \cdots y_{n-1}^{\frac{i_{n-1}\alpha}{\gamma_{n-1}}} \left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{i_n\alpha}{\gamma_n}}\right]$$

$$=\Gamma(n)\int\ldots\int y_1^{\frac{i_1\alpha}{\gamma_1}} y_2^{\frac{i_2\alpha}{\gamma_2}} \cdots y_{n-1}^{\frac{i_{n-1}\alpha}{\gamma_{n-1}}} \left(1-y_1-y_2-\cdots-y_{n-1}\right)^{\frac{i_n\alpha}{\gamma_n}} dy_1\cdots dy_{n-1}$$

$$=\frac{\Gamma(n)\Gamma\left(\frac{i_1\alpha}{\gamma_1}+1\right)\Gamma\left(\frac{i_2\alpha}{\gamma_2}+1\right)\cdots\Gamma\left(\frac{i_n\alpha}{\gamma_n}+1\right)}{\Gamma\left(\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+n\right)}$$

which is the Dirichlet integral (Rao 1954)

For $E\left[y_n^{\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots\frac{i_n}{\gamma_n}}\right]$, let $\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}=c$, then

$$E\left[y_n^{\,c}\right]$$

$$=\frac{(-1)^n}{\Gamma(n)}\int_0^\infty y_n^{\,c}\sum_{i=1}^{P(n)}\left\{\alpha^{-1}y_n^{\,k_i-1}(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right\}\exp(-y_n)dy_n$$

$$=\frac{(-1)^n}{\Gamma(n)}\sum_{i=1}^{P(n)}\left\{\left[\alpha^{-1}(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\right]\int_0^\infty y_n^{\,c+k_i-1}\exp(-y_n)dy_n\right\}$$

$$=\frac{(-1)^n}{\Gamma(n)}\sum_{i=1}^{P(n)}\left[\alpha^{-1}(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\Gamma(c+k_i)\right]$$

$$=\frac{(-1)^n}{\Gamma(n)}\sum_{i=1}^{P(n)}\left[\alpha^{-1}(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\Gamma(c)c^{\overline{k_i}}\right] \qquad (8)$$

where $c^{\overline{k_i}}$ is the rising factorial defined as $c(c+1)\cdots(c+k_i-1)$ by Knuth (1992).

Considering $P_s(n,i)$ in the above equation, it is the total number of set partitions corresponding to the i^{th} partition of *n* such that $n_1+n_2+\cdots+n_{k_i}=n$, $n_1,n_2,\ldots,n_{k_i}>0$.

It has been shown by McCullagh and Wilks (1988) that

$$P_s(n,i)=\frac{n!}{n_1!n_2!\cdots n_{k_i}!m_1!m_2!\cdots m_d!}$$

where $m_1,m_2,\ldots,m_d$ are the number of each distinct summand.

Then, the product $P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)$

$$=\frac{n!}{n_1!n_2!\cdots n_{k_i}!m_1!m_2!\cdots m_d!}\alpha^{\underline{n_1}}\alpha^{\underline{n_2}}\cdots\alpha^{\underline{n_{k_i}}}$$

$$=\frac{n!}{m_1!m_2!\cdots m_d!}\frac{\alpha!}{n_1!(\alpha-n_1)!}\frac{\alpha!}{n_2!(\alpha-n_2)!}\cdots\frac{\alpha!}{n_k!\left(\alpha-n_{k_i}\right)!}$$

$$=\frac{n!}{m_1!m_2!\cdots m_d!}\binom{\alpha}{n_1}\binom{\alpha}{n_2}\cdots\binom{\alpha}{n_{k_i}}$$

$$=\frac{n!}{k_i!}\frac{k_i!}{m_1!m_2!\cdots m_d!}\binom{\alpha}{n_1}\binom{\alpha}{n_2}\cdots\binom{\alpha}{n_{k_i}},$$

where $\frac{k_i!}{m_1!m_2!\cdots m_d!}$ is the number of permutations of $n_1,n_2,\cdots,n_{k_i}$ of every possible order.

When sum $P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)$ over the same value of k_i, say, *k*, then,

$$\sum_{k_i=k}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)$$

$$=\sum_{k_i=k}\left[\frac{n!}{k_i!}\frac{k_i!}{m_1!m_2!\cdots m_d!}\binom{\alpha}{n_1}\binom{\alpha}{n_2}\cdots\binom{\alpha}{n_{k_i}}\right]$$

$$=\frac{n!}{k!}\sum_{k_i=k}\left[\frac{k!}{m_1!m_2!\cdots m_d!}\binom{\alpha}{n_1}\binom{\alpha}{n_2}\cdots\binom{\alpha}{n_k}\right]$$

which equals $C(n,k,\alpha)$, the C-numbers defined by Charalambides (1977). Note that the summation is over all the permutations of n_{k_i} with $k_i=k$.

Using the equality $(-1)^{k_i}c^{\overline{k_i}}=(-c)^{\underline{k_i}}$ (Goldman, Joichi, Reiner and White 1976),

$$E\left[y_n{}^c\right]$$

$$=\frac{(-1)^n}{\Gamma(n)}\sum_{i=1}^{P(n)}\left[\alpha^{-1}(-1)^{k_i}P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)\Gamma(c)c^{\overline{k_i}}\right]$$

$$=\frac{(-1)^n}{\Gamma(n)}\alpha^{-1}\Gamma(c)\left\{\sum_{i=1}^{P(n)}\left[P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)(-c)^{\underline{k_i}}\right]\right\}$$

$$=\frac{(-1)^n}{\Gamma(n)}\alpha^{-1}\Gamma(c)\left\{\sum_{k=1}^{n}\left[\sum_{k_i=k}\left(P_s(n,i)\left(\prod_{j=1}^{k_i}\alpha^{\underline{n_j}}\right)(-c)^{\underline{k_i}}\right)\right]\right\}$$

$$=\frac{(-1)^n}{\Gamma(n)}\alpha^{-1}\Gamma(c)\sum_{k=1}^{n}\left[C(n,k,\alpha)(-c)^{\underline{k}}\right]$$

$$=\frac{(-1)^n}{\Gamma(n)}\alpha^{-1}\Gamma(c)(-\alpha c)^{\underline{n}}$$ (using equation 1.3 by Charalambides 1977)

$$=\frac{(-1)^n}{\Gamma(n)}\alpha^{-1}\Gamma(c)(-1)^n(\alpha c)^{\overline{n}}$$ (using the formula by Goldman *et al*. 1976)

$$=\frac{1}{\Gamma(n)}(\alpha c+1)(\alpha c+2)\cdots(\alpha c+n-1)\Gamma(c+1).$$

Therefore, the general moment of $x_1, x_2, \cdots, x_n$ is

$$E\left[x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\right]$$

$$=\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}E\left[y_1^{\frac{i_1\alpha}{\gamma_1}}y_2^{\frac{i_2\alpha}{\gamma_2}}\cdots y_{n-1}^{\frac{i_{n-1}\alpha}{\gamma_{n-1}}}(1-y_1-y_2-\cdots-y_{n-1})^{\frac{i_n\alpha}{\gamma_n}}\right]E\left[y_n^{\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots\frac{i_n}{\gamma_n}}\right]$$

$$=\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}\frac{\Gamma(n)\Gamma\left(\frac{i_1\alpha}{\gamma_1}+1\right)\Gamma\left(\frac{i_2\alpha}{\gamma_2}+1\right)\cdots\Gamma\left(\frac{i_n\alpha}{\gamma_n}+1\right)}{\Gamma\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+n\right]}$$

$$\cdot\frac{1}{\Gamma(n)}\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+1\right]\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+2\right]\cdots$$

$$\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+(n-1)\right]\Gamma\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+1\right]$$

$$=\frac{\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}\Gamma\left(\frac{i_1\alpha}{\gamma_1}+1\right)\Gamma\left(\frac{i_2\alpha}{\gamma_2}+1\right)\cdots\Gamma\left(\frac{i_n\alpha}{\gamma_n}+1\right)\Gamma\left[\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+1\right]}{\Gamma\left[\alpha\left(\frac{i_1}{\gamma_1}+\frac{i_2}{\gamma_2}+\cdots+\frac{i_n}{\gamma_n}\right)+1\right]} \quad (9)$$

From the general moment, the expectation and the variance of any random variable, and the covariance and the correlation coefficient of any pair of variables can be derived. However, as Lu and Bhattacharyya (1990) pointed out, the correlation coefficient is non-negative.

4. Application

For illustration purpose, we fit the proposed model mode with n=3 to the data of Campbell and McCabe (1984). The data contains GPA, SAT (both math ban verbal) of 234 freshmen that majored in computer science. The data was downloaded from www.math.aau.dk/~s0ren/BusinessStatistics/data.html. Campbell and McCabe (1984) had shown that GPA is positively correlated with SAT's math score and verbal score. Let GPA be X_1, SAT math score be X_2, and SAT verbal score be X_3. The PDF of the trivariate Weibull is

$$\begin{aligned}
& f(x_1,x_2,x_3) \\
&= (-1)^3 \frac{\partial^3 S(x_1,x_2,x_3)}{\partial x_1 \partial x_2 \partial x_3} \\
&= \frac{\gamma_1\gamma_2\gamma_3}{\alpha^2 x_1 x_2 x_3} exp\left(-\left(\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha}\right) \\
&\times \left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}} \left(\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{-3+\alpha} \\
&\times \left(2 + 3\alpha\left(-1 + \left(\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha}\right)\right. \\
&\left. + \alpha^2\left(1 - 3\left(\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha} + \left(\left(\frac{x_1}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_2}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_3}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{2\alpha}\right)\right).
\end{aligned}$$

Let n denote the number of observations, then, the log-likelihood function becomes

$$\begin{aligned}
& \sum_{i=1}^{n} \log\left(f\left(x_{1_i}, x_{2_i}, x_{3_i}\right)\right) \\
&= \sum_{i=1}^{n} \log\left(\frac{\gamma_1\gamma_2\gamma_3}{\alpha^2 x_{1_i} x_{2_i} x_{3_i}}\right) - \sum_{i=1}^{n}\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}} + \left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha}
\end{aligned}$$

$$+\sum_{i=1}^{n}\log\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}\left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}\left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+\left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{-3+\alpha}\right)$$

$$+\sum_{i=1}^{n}\log\left(2+3\alpha\left(-1+\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+\left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha}\right)\right.$$

$$\left.+\alpha^2\left(1-3\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+\left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{\alpha}+\left(\left(\frac{x_{1_i}}{\lambda_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{x_{2_i}}{\lambda_2}\right)^{\frac{\gamma_2}{\alpha}}+\left(\frac{x_{3_i}}{\lambda_3}\right)^{\frac{\gamma_3}{\alpha}}\right)^{2\alpha}\right)\right)$$

The parameters are estimated by maximizing the likelihood function, and the standard deviation for 95% confidence interval are approximated using the inverse of the negative Hessian of the log-likelihood function. The results of the fitted trivariate Weibull model are in Table 1.

After the parameter estimates are obtained, the covariance (denoted by *Cov*) and the variance (denoted by *Var*) can be calculated using the general moment of Equation 9.

The correlation coefficient between X_1 and X_2 is

$$\frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}}$$

$$=\frac{E\left(X_1^{\,1}X_2^{\,1}X_3^{\,0}\right)-E\left(X_1^{\,1}X_2^{\,0}X_3^{\,0}\right)E\left(X_1^{\,0}X_2^{\,1}X_3^{\,0}\right)}{\sqrt{E\left(X_1^{\,2}X_2^{\,0}X_3^{\,0}\right)-E^2\left(X_1^{\,1}X_2^{\,0}X_3^{\,0}\right)}\sqrt{E\left(X_1^{\,0}X_2^{\,2}X_3^{\,0}\right)-E^2\left(X_1^{\,0}X_2^{\,1}X_3^{\,0}\right)}}$$

=0.262.

Repeatedly, the correlation coefficient between X_1 and X_3, X_2 and X_3 are calculated. Table 2 shows the correlation confidents and their 95% confidence intervals of the 3 pairs of the random variables. The confidence intervals are calculated using Fisher's *z* transformation formulas (Krishnamoorthy andXia 2007).

Note that when α equals 1, X_1, X_2 and X_3 are independent. The estimate of α in this example is 0.843 which implies that the 3 random variables are not much correlated. In fact, the three correlation coefficients are between 0.2 and 0.4 which indicate that each pair of the random variables has low correlation suggested by Guldford's interpretation of correlation coefficient (Cukier andPanjwani 2007). Therefore, there exists low correlation between any pair of GAP, SAT math score, and SAT verbal score.

Table 1

Parameter	Estimate	95% Confidence Interval
α	0.843	(0.783, 0.903)
λ_1	3.029	(2.937, 3.122)
γ_1	4.381	(3.910, 4.852)
λ_2	629.819	(619.866, 639.772)
γ_2	8.442	(7.624, 9.259)
λ_3	548.301	(535.404, 561.198)
γ_3	5.697	(5.168, 6.225)

Table 2

	Estimate	95% Confidence Interval
$Cov(X_1, X_2)$	0.262	(0.202, 0.321)
$Cov(X_1, X_3)$	0.259	(0.198, 0.318)
$Cov(X_2, X_3)$	0.267	(0.206, 0.325)

5. Conclusions

In this paper, we proposed a multivariate Weibull model that is similar to the genuine multivariate Weibull one by Crowder (1989). We further derived the explicit form of PDF, CDF, and the general moment. The explicit forms of these functions are desirable for numerical computations. We believe that the proposed model is a candidate for, first, testing the overall association on a given number of random variables that are Weibull distributed. It also becomes a competing risks model when the random variables in the proposed model are survival times to the corresponding events that, in business and management, could be paying off a loan, defaulting a loan, leaving a company, etc.

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