

Approximate Bayes Estimators of the Logistic Distribution Parameters Based on Progressive Type-II Censoring Scheme

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Abstract

In this paper we develop approximate Bayes estimators of the parameters, reliability, and hazard rate functions of the Logistic distribution by using Lindley's approximation, based on progressively type-II censoring samples. Non-informative prior distributions are used for the parameters. Quadratic, linex and general Entropy loss functions are used. The statistical performances of the Bayes estimates relative to quadratic, linex and general entropy loss functions are compared to those of the maximum likelihood based on simulation study.

Keywords: Logistic distribution, progressively type-II censoring, linex and general Entropy loss functions, Lindley's Bayes approximation.

2010 AMS Classification: 62F10, 62F15, 62N01, 62N05.

1. Introduction

The logistic function is one of the most popular and widely used for growth models in demographic studies and proposed by Verhulst (1838-1845) see Balakrishnan (1992). The normal distribution resembles to logistic distribution in shape but the logistic distribution has thicker tails and higher kurtosis than the normal distribution. The logistic distribution has been applied in studies of population growth, physicochemical phenomena, bio-assay and a life test data, see Balakrishnan (1992), and of biochemical data by Gupta et al. (1967). Oliver (1964) used the logistic function as a model for agricultural production data. Scerri and Farrugia(1996) compared between the logistic distribution and weibull distribution for modeling wind speed data. Subrata et al. (2012) proposed askew logistic distribution and derived some properties for this distribution. Many researchers have used asymmetric loss function applied to several statistical models (Bekker et al. (2000), Calabria and Pulcini (1996), Wen and Levy (2001) and Dey et al. (1987)). Censoring is a common phenomenon in life-testing and reliability studies. The experimenter may be unable to obtain complete information on failure times for all experimental units. For example, individuals in a clinical trial may withdraw from the

study, or the study may have to be terminated for lack of funds. In an industrial experiment, units may break accidentally. In many situations, however, the removal of units prior to failure is preplanned in order to provide savings in terms of time and cost associated with testing. Progressive Type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides before hand the quantity m , the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-R_1-2$ units are randomly removed from the experiment. Finally, at the time of the m -th failure, all the remaining surviving units $R_m = n-m-R_1-\dots-R_{m-1}$ are removed from the experiment.

Therefore, a progressive Type-II censoring scheme consists of m , and $R_1, \dots, R_m$, such that $R_1 + \dots + R_m = n-m$. The m failure times obtained from a progressive Type-II censoring scheme will be denoted by $x_1, \dots, x_m$.

2. Maximum Likelihood Estimators (MLEs)

In this section, we derive the MLEs of the unknown parameters based on progressively type-II censoring samples. Assume the failure time distribution to be the logistic distribution with probability density function (pdf)

$$f(x; \mu, \beta) = \frac{e^{-\frac{(x-\mu)}{\beta}}}{\beta \left(1 + e^{-\frac{(x-\mu)}{\beta}} \right)^2}; -\infty < x < \infty, -\infty < \mu < \infty, \beta > 0, \quad (2.1)$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x; \mu, \beta) = \frac{1}{1 + e^{-\frac{(x-\mu)}{\beta}}}. \quad (2.2)$$

Based on the observed sample $x_1 < \dots < x_m$ from a progressive type-II censoring scheme, $(R_1, \dots, R_m)$, the likelihood function can be written as

$$L(\underline{x}; \mu, \beta) = c \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{R_i}, \quad (2.3)$$

where $c = n(n-1-R_1)\dots(n-R_1-\dots-R_{m-1}-m+1)$, $f(\cdot)$ and $F(\cdot)$ are given by (2.1) and (2.2) respectively. Then

$$L(\underline{x}; \mu, \beta) = \frac{c}{\beta^m} e^{-\sum_{i=1}^m \frac{(x_i - \mu)(R_i + 1)}{\beta}} \prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)}$$

The log-likelihood function can be written as

$$\begin{aligned} \text{LogL} = \ell &= \text{Logc} - m\text{Log}\beta - \frac{1}{\beta} \sum_{i=1}^m (R_i + 1)(x_i - \mu) + \log \left[\prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)} \right] \\ \ell &= \text{Logc} - m\text{Log}\beta - \frac{1}{\beta} \sum_{i=1}^m (R_i + 1)(x_i - \mu) - \sum_{i=1}^m (R_i + 2) \text{Log} \left[1 + e^{-\frac{(x_i - \mu)}{\beta}} \right] \end{aligned} \quad (2.4)$$

The MLEs of the unknown parameters can be obtained by differentiating the log-likelihood function (2.4) with respect to the unknown parameters and equating to zero, we get

$$\left. \begin{aligned} \frac{\sum_{i=1}^m (R_i + 1)}{\hat{\beta}} - \sum_{i=1}^m \frac{e^{-\frac{(x_i - \hat{\mu})}{\hat{\beta}}} (R_i + 2)}{\left(1 + e^{-\frac{(x_i - \hat{\mu})}{\hat{\beta}}} \right) \hat{\beta}} &= 0, \\ -\frac{m}{\hat{\beta}} + \frac{\sum_{i=1}^m (R_i + 1)(x_i - \hat{\mu})}{\hat{\beta}^2} - \sum_{i=1}^m \frac{e^{-\frac{(x_i - \hat{\mu})}{\hat{\beta}}} (R_i + 2)(x_i - \hat{\mu})}{\left(1 + e^{-\frac{(x_i - \hat{\mu})}{\hat{\beta}}} \right) \hat{\beta}^2} &= 0. \end{aligned} \right\} \quad (2.5)$$

The solution of the non-linear equations (2.5) is $\hat{\mu}, \hat{\beta}$.

The MLEs of the reliability function, and the hazard rate function are given as

$$\hat{R}(t) = \frac{1}{1 + e^{\frac{t - \hat{\mu}}{\hat{\beta}}}}, \quad \hat{H}(t) = \frac{1}{\hat{\beta} \left(1 + e^{-\frac{(t - \hat{\mu})}{\hat{\beta}}} \right)}.$$

3. Bayes Estimates for the Unknown Parameters μ and β

In this section Bayesian estimation of the parameters of the logistic distribution along with reliability function and hazard rate function, using progressive type-II censoring samples, based on the square error loss function, linear-exponential loss function, and general Entropy loss function are obtained.

Assuming that μ and β are independent random variables, and no information about μ and β is available, considering a non-informative prior distribution for β in the form

$$\pi(\beta) \propto \frac{1}{\beta}; 0 < \beta < \infty,$$

and a non-informative prior distribution for μ in the form

$$\pi(\mu) \propto k; -\infty < \mu < \infty, K : \text{constant}.$$

The joint prior distribution for μ and β is given by

$$\pi(\mu, \beta) \propto \frac{1}{\beta}; 0 < \beta < \infty, -\infty < \mu < \infty. \quad (3.1)$$

by using equations (2.3, 3.1) we get the joint posterior distribution for μ and β as follows

$$\begin{aligned} \pi(\mu, \beta | \underline{x}) &= \frac{\pi(\mu, \beta) L(\underline{x} | \mu, \beta)}{\int_0^\infty \int_{-\infty}^\infty \pi(\mu, \beta) L(\underline{x} | \mu, \beta) d\mu d\beta} \\ &= \frac{\frac{c}{\beta^{m+1}} e^{-\sum_{i=1}^m \frac{(x_i - \mu)(R_i + 1)}{\beta}} \prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)}}{\int_0^\infty \int_{-\infty}^\infty \frac{c}{\beta^{m+1}} e^{-\sum_{i=1}^m \frac{(x_i - \mu)(R_i + 1)}{\beta}} \prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)} d\mu d\beta}. \end{aligned} \quad (3.2)$$

Integration in equation (3.2) cannot be obtained in a closed form, so we solve it numerically. In the following sections we derive Bayesian estimators for location and scale parameters, the reliability function, and the hazard rate function under some loss functions.

3.1 Bayesian Estimators Under Square Error Loss Function

1. Bayesian estimator for location parameter μ

$$\hat{\mu}_{sq} = E(\mu) = \int_0^\infty \int_{-\infty}^\infty \mu \frac{\frac{c}{\beta^{m+1}} e^{-\sum_{i=1}^m \frac{(x_i - \mu)(R_i + 1)}{\beta}} \prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)}}{\int_0^\infty \int_{-\infty}^\infty \frac{c}{\beta^{m+1}} e^{-\sum_{i=1}^m \frac{(x_i - \mu)(R_i + 1)}{\beta}} \prod_{i=1}^m \left(1 + e^{-\frac{(x_i - \mu)}{\beta}} \right)^{-(R_i + 2)} d\mu d\beta} d\mu d\beta. \quad (3.3)$$

Provided That $E(\mu)$ exists and is finite. This integration cannot be solved analytically, so we use Lindley's Bayes approximation, Lindley (1980). Let $u(\mu, \beta)$ be a function of μ and β , and we want to find Bayes estimator for it, based on $\pi(\mu, \beta)$ as a prior distribution. The Log-likelihood function for the logistic distribution based on

progressive type II censored samples is given by (2.4), Bayes estimate of $u(\mu, \beta)$ using Lindley approximation is obtained as follows:

$$E(u(\mu, \beta) | \underline{x}) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\mu, \beta) \pi(\mu, \beta) L(\underline{x} | \mu, \beta) d\mu d\beta}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(\mu, \beta) L(\underline{x} | \mu, \beta) d\mu d\beta}.$$

Let $Q(\mu, \beta) = \log[\pi(\mu, \beta)]$

$$E(u(\mu, \beta) | \underline{x}) \approx \left(u(\mu, \beta) + \frac{1}{2} \left[\sum_i \sum_j (u_{ij} + 2u_i Q_j) \tau_{ij} + \sum_i \sum_j \sum_k \sum_w L_{ijk} u_w \tau_{ij} \tau_{kw} \right] \right)_{(\mu, \beta)_{ML}} \quad (3.4)$$

$$\begin{aligned} \forall i, j, k, w = 1, 2, Q_1 &= \frac{\partial Q(\mu, \beta)}{\partial \mu}, Q_2 = \frac{\partial Q(\mu, \beta)}{\partial \beta}, u_1 = \frac{\partial u(\mu, \beta)}{\partial \mu}, \\ u_2 &= \frac{\partial u(\mu, \beta)}{\partial \beta}, u_{11} = \frac{\partial^2 u(\mu, \beta)}{\partial \mu^2}, u_{22} = \frac{\partial^2 u(\mu, \beta)}{\partial \beta^2}, u_{12} = \frac{\partial^2 u(\mu, \beta)}{\partial \mu \partial \beta}, \\ L_{11} &= \frac{\partial^2 \ell}{\partial \mu^2}, L_{12} = \frac{\partial^2 \ell}{\partial \mu \partial \beta}, L_{22} = \frac{\partial^2 \ell}{\partial \beta^2}, L_{111} = \frac{\partial^3 \ell}{\partial \mu^3}, L_{112} = \frac{\partial^3 \ell}{\partial \mu^2 \partial \beta}, L_{122} = \frac{\partial^3 \ell}{\partial \mu \partial \beta^2}, L_{222} = \frac{\partial^3 \ell}{\partial \beta^3}. \end{aligned}$$

Calculate the elements of matrix $\{-L_{ij}\}$

$$\Sigma = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \mu^2} & -\frac{\partial^2 \ell}{\partial \mu \partial \beta} \\ -\frac{\partial^2 \ell}{\partial \mu \partial \beta} & -\frac{\partial^2 \ell}{\partial \beta^2} \end{bmatrix}^{-1},$$

by using Mathematica program we can calculate the inverse matrix, and find the values of τ_{ij} .

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = \mu$, the Bayesian estimator for location parameter μ is given as

$$\hat{\mu}_{sq} \square \mu - \frac{\tau_{12}}{\beta} + \frac{1}{2} \left[L_{111} \tau_{11}^2 + 3L_{112} \tau_{12} \tau_{11} + L_{122} (\tau_{22} \tau_{11} + 2\tau_{12}^2) + L_{222} \tau_{22} \tau_{12} \right]$$

2. Bayesian estimator for scale parameter β

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = \beta$, the Bayesian estimator for scale parameter β is given as

$$\hat{\beta}_{sq} \square \beta - \frac{\tau_{22}}{\beta} + \frac{1}{2} \left[L_{111} \tau_{11} \tau_{12} + L_{112} (\tau_{11} \tau_{22} + 2\tau_{12}^2) + 3L_{122} \tau_{22} \tau_{12} + L_{222} \tau_{22}^2 \right]$$

3. Bayesian estimator for reliability function $R(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = R(t)$, the Bayesian estimator for reliability function $R(t)$ is given by

$$\hat{R}_{sq} \square R(t) - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2} \left[\frac{u_{11}\tau_{11} + 2u_{21}\tau_{12}}{+u_{22}\tau_{22}} \right] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)} \right]$$

4. Bayesian estimator for hazard rate function $H(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = H(t)$, the Bayesian estimator for hazard rate function $H(t)$ is given by

$$\hat{H}_{sq} \square H(t) - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2} \left[\frac{u_{11}\tau_{11} + 2u_{21}\tau_{12}}{+u_{22}\tau_{22}} \right] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)} \right]$$

3.2 Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

1. Bayesian estimator for location parameter μ

$$\hat{\mu}_{LINEX} = -\frac{1}{c} \log[E(e^{-c\mu})]$$

Provided That $E(e^{-c\mu})$ exists and is finite. Substitution in equation (3.4) ,

$Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = e^{-c\mu}$, the Bayesian estimator for location parameter μ is given as

$$\hat{\mu}_{LINEX} \square -\frac{1}{c} \log \left[e^{-c\mu} + \frac{ce^{-c\mu}\tau_{12}}{\beta} + \frac{c^2e^{-c\mu}\tau_{11}}{2} - \frac{1}{2} \left[\frac{ce^{-c\mu}L_{111}\tau_{11}^2 + 3ce^{-c\mu}L_{112}\tau_{12}\tau_{11} + ce^{-c\mu}L_{122}(\tau_{22}\tau_{11} + 2\tau_{12}^2) + ce^{-c\mu}L_{222}\tau_{22}\tau_{12}}{ce^{-c\mu}L_{222}\tau_{22}\tau_{12}} \right] \right]$$

2. Bayesian estimator for scale parameter β

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = e^{-c\beta}$, the Bayesian estimator for scale parameter β is given as

$$\hat{\beta}_{LINEX} = -\frac{1}{c} \log \left[e^{-c\beta} + \frac{ce^{-c\beta}\tau_{22}}{\beta} + \frac{c^2e^{-c\beta}\tau_{22}}{2} - \frac{1}{2} \left[\frac{ce^{-c\beta}L_{111}\tau_{11}\tau_{12} + ce^{-c\beta}L_{112}(\tau_{22}\tau_{11} + 2\tau_{12}^2) + 3ce^{-c\beta}L_{122}\tau_{12}\tau_{22} + ce^{-c\beta}L_{222}\tau_{22}^2}{\beta} \right] \right]$$

3. Bayesian estimator for reliability function $R(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = e^{-cR(t)}$, the Bayesian estimator for reliability function $R(t)$ is given by

$$\hat{R}_{LINEX} = -\frac{1}{c} \log \left[e^{-cR(t)} - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2}[u_{11}\tau_{11} + 2u_{21}\tau_{12} + u_{22}\tau_{22}] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{\beta} \right] \right]$$

4. Bayesian estimator for hazard rate function $H(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = -\frac{1}{\beta}, u = e^{-cH(t)}$, the Bayesian estimator for hazard rate function $H(t)$ is given by

$$\hat{H}_{LINEX} = -\frac{1}{c} \log \left[e^{-cH(t)} - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2}[u_{11}\tau_{11} + 2u_{21}\tau_{12} + u_{22}\tau_{22}] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{\beta} \right] \right]$$

3.3 Bayesian Estimators Under General Entropy Loss Function

1. Bayesian estimator for location parameter μ

$$\hat{\mu}_{Entropy} = \left[E(\mu^{-q}) \right]^{-\frac{1}{q}}$$

Provided That $E(\mu^{-q})$ exists and is finite. Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = \mu^{-q}$, the Bayesian estimator for location parameter μ is given as

$$\hat{\mu}_{Gentropy} \square \left[\mu^{-q} + \frac{q\mu^{-q-1}\tau_{12}}{\beta} + \frac{q(q+1)\mu^{-q-2}\tau_{11}}{2} - \frac{1}{2} \left[\frac{q\mu^{-q-1}L_{111}\tau_{11}^2 + 3q\mu^{-q-1}L_{112}\tau_{12}\tau_{11} + L_{122}(q\mu^{-q-1}(\tau_{22}\tau_{11} + 2\tau_{12}^2)) + q\mu^{-q-1}L_{222}\tau_{22}\tau_{12}}{q\mu^{-q-1}L_{111}\tau_{11}\tau_{12} + q\mu^{-q-1}L_{112}(\tau_{22}\tau_{11} + 2\tau_{12}^2) + 3q\mu^{-q-1}L_{122}\tau_{22}\tau_{12} + q\mu^{-q-1}L_{222}\tau_{22}^2} \right] \right]^{-\frac{1}{q}}$$

2. Bayesian estimator for scale parameter β

Substitution in equation (3.4), $Q_1 = 0, Q_2 = \frac{-1}{\beta}, u = \beta^{-q}$, the Bayesian estimator for scale parameter β is given as

$$\hat{\beta}_{Gentropy} \square \left[\beta^{-q} + \frac{q\beta^{-q-1}\tau_{22}}{\beta} + \frac{q(q+1)\beta^{-q-2}\tau_{22}}{2} - \frac{1}{2} \left[\frac{q\beta^{-q-1}L_{111}\tau_{11}\tau_{12} + q\beta^{-q-1}L_{112}(\tau_{22}\tau_{11} + 2\tau_{12}^2) + 3q\beta^{-q-1}L_{122}\tau_{22}\tau_{12} + q\beta^{-q-1}L_{222}\tau_{22}^2}{q\beta^{-q-1}L_{111}\tau_{11}\tau_{12} + q\beta^{-q-1}L_{112}(\tau_{22}\tau_{11} + 2\tau_{12}^2) + 3q\beta^{-q-1}L_{122}\tau_{22}\tau_{12} + q\beta^{-q-1}L_{222}\tau_{22}^2} \right] \right]^{-\frac{1}{q}}$$

3. Bayesian estimator for reliability function $R(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = -\frac{1}{\beta}, u = (R(t))^{-q}$, the Bayesian estimator for reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy} \square \left[(R(t))^{-q} - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2}[u_{11}\tau_{11} + 2u_{21}\tau_{12} + u_{22}\tau_{22}] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)} \right] \right]^{-\frac{1}{q}}$$

4. Bayesian estimator for hazard rate function $H(t)$

Substitution in equation (3.4), $Q_1 = 0, Q_2 = -\frac{1}{\beta}, u = (H(t))^{-q}$, the Bayesian estimator for hazard rate function $H(t)$ is given by

$$\hat{H}_{Gentropy} \square \left[(H(t))^{-q} - \frac{1}{\beta}(u_1\tau_{12} + u_2\tau_{22}) + \frac{1}{2}[u_{11}\tau_{11} + 2u_{21}\tau_{12} + u_{22}\tau_{22}] + \frac{1}{2} \left[\frac{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)}{L_{111}(u_1\tau_{11}^2 + u_2\tau_{11}\tau_{12}) + L_{112}(u_2(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3u_1\tau_{21}\tau_{11}) + L_{122}(u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 3u_2\tau_{12}\tau_{22}) + L_{222}(u_1\tau_{22}\tau_{21} + u_2\tau_{22}^2)} \right] \right]^{-\frac{1}{q}}$$

It is worth noting that when the value $q = -1$, the general entropy loss function is the same as the squared error loss function.

4. Simulation Studies

To demonstrate the importance of the results obtained in the preceding sections, simulation studies are conducted. For this purpose, by using Monte Carlo method, with fixed sample size n (the total items put in a life test), with constant censoring scheme, where $R_1 = R_2 = R_3 = \dots = R_m$, where m is the sample size of progressively censored from the sample of size n .

The following algorithm is used to generate sample based on progressive type-II censoring scheme, based on any continuous df F , see Balakrishnan and Aggarwala (2000).

1. Generate m independent Uniform (0,1) observations $W_1, \dots, W_m$.
2. Set $V_i = W_i^{1/\gamma_i}$, $\gamma_i = \left(i + \sum_{j=m-i+1}^m R_j \right)$ for $i = 1, 2, \dots, m$.
3. $U_i = 1 - V_m V_{m-1} \dots V_{m-i+1}$, $i = 1, 2, \dots, m$.
4. Set $X_i = F^{-1}(U_i)$, then X_i , for $i = 1, 2, \dots, m$, is the progressive type-II censoring scheme based on the df F .
5. We repeated steps 1,2,3 and 4 (1000) times, for different values of n and m .

estimation average = $\frac{\sum_{i=1}^{1000} \hat{\theta}_i}{1000}$, mean square error = $\frac{\sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2}{1000}$, where, θ is the parameter and $\hat{\theta}$ is the estimator.

All the computations are prepared by Mathematica 9.

Since the non-linear equations (2.5) are not solvable analytically, numerical methods can be used, as Newton Rhapsod method with initial values closed to real values of the parameters.

Throughout this section we will use the following abbreviations:

- ML : means that the estimate by using the (MLE),
- B_{Sq} : means that the estimate under squared error loss function,
- $B_{Lx, c=16}$: means that the estimate under linex loss function at $c = 16$,
- $B_{Lx, c=18}$: means that the estimate under linex loss function at $c = 18$,
- $B_{Lx, c=20}$: means that the estimate under linex loss function at $c = 20$,
- $B_{Ge, q=5}$: means that the estimate under general entropy loss function at $q = 5$,

$B_{Ge,q=7}$: means that the estimate under general entropy loss function at $q = 7$,

$B_{Ge,q=10}$: means that the estimate under general entropy loss function at $q = 10$.

Table1: The average, mean square error, when n=200, m=100, scheme (100*1) and $\mu=0$.

$B_{Lx,c=20}$	$B_{Lx,c=18}$	$B_{Lx,c=16}$	$B_{Ge,q=10}$	$B_{Ge,q=7}$	$B_{Ge,q=5}$	B_{Sq}	ML	β
<i>The average, (mean square error) of the estimators of parameter μ</i>								
-0.5513 (0.3122)	-0.5625 (0.326)	-0.5748 (0.343)	0.1296 (0.016)	0.1127 (0.009)	0.0990 (0.004)	-0.1699 (0.0289)	-0.1631 (1.1059)	0.7
-0.8668 (0.7577)	-0.8754 (0.774)	-0.8846 (0.792)	0.2847 (0.081)	0.2560 (0.050)	0.2320 (0.025)	-0.3443 (0.1185)	-0.3419 (2.2922)	0.8
-1.0092 (1.0254)	-1.0175 (1.0438)	-1.0264 (1.0641)	0.3497 (0.1223)	0.3154 (0.0764)	0.2864 (0.0387)	-0.4184 (0.1751)	-0.4147 (3.2864)	0.9
<i>The average, (mean square error) of the estimators of parameter β</i>								
0.6687 (0.0010)	0.6710 (0.0008)	0.6736 (0.0007)	0.6734 (0.0007)	0.6796 (0.0004)	0.6840 (0.0003)	0.6969 (0.0001)	0.6797 (0.0142)	0.7
0.7396 (0.0037)	0.7421 (0.0034)	0.7448 (0.0031)	0.7469 (0.0028)	0.7534 (0.0022)	0.7582 (0.0018)	0.7723 (0.0008)	0.756 (0.0352)	0.8
0.8289 (0.0051)	0.8315 (0.0047)	0.8345 (0.0043)	0.8399 (0.0036)	0.8471 (0.0028)	0.8523 (0.0023)	0.8680 (0.0011)	0.8505 (0.0446)	0.9
<i>The average, (mean square error) of the estimators of reliability function $R(t)$</i>								
$R(t=2)=0.0543$								0.7
0.0565 (8×10^{-6})	0.0565 (8×10^{-6})	0.0566 (8×10^{-6})	0.0346 (0.0004)	0.0363 (0.0003)	0.0395 (0.0002)	0.0567 (8×10^{-6})	0.0527 (0.00009)	
$R(t=2)=0.0759$								0.8
0.0754 (3×10^{-6})	0.0754 (3×10^{-6})	0.0754 (2×10^{-6})	0.0388 (0.0014)	0.0413 (0.0012)	0.0463 (0.0009)	0.0749 (2×10^{-6})	0.0717 (0.0003)	
$R(t=2)=0.0978$								0.9
0.0962 (6×10^{-6})	0.0963 (6×10^{-6})	0.0963 (5×10^{-6})	0.0506 (0.0022)	0.0543 (0.0019)	0.0613 (0.0013)	0.0958 (5×10^{-6})	0.0924 (0.0005)	
<i>The average, (mean square error) of the estimators of hazard rate function $H(t)$</i>								
$H(t=2)=1.3510$								0.7
1.3465 (0.0001)	1.3498 (0.0002)	1.3552 (0.0002)	1.4751 (0.0167)	1.6282 (0.0802)	1.9055 (0.3172)	(2×10^{13}) (2×10^{26})	(2×10^{13}) (2×10^{28})	
$H(t=2)=1.1552$								0.8
1.2116 (0.0033)	1.2164 (0.0039)	1.2238 (0.0049)	1.3319 (0.0321)	1.478 (0.1066)	1.7434 (0.3531)	(5×10^{13}) (3×10^{27})	(5×10^{13}) (2×10^{29})	
$H(t=2)=1.0025$								0.9
1.0683 (0.0044)	1.0744 (0.0053)	1.0832 (0.0067)	1.1686 (0.0283)	1.2975 (0.0889)	1.5320 (0.2859)	(3×10^{13}) (8×10^{26})	(3×10^{13}) (3×10^{28})	

Table 2: The average, mean square error, when $n=100$, $m=50$, scheme (50*1) and $\mu = 0$.

$B_{Lx, c=20}$	$B_{Lx, c=18}$	$B_{Lx, c=16}$	$B_{Ge, q=10}$	$B_{Ge, q=7}$	$B_{Ge, q=5}$	B_{Sq}	ML	β
<i>The average, (mean square error) of the estimators of parameter μ</i>								
-1.7937 (3.2202)	-1.8249 (3.3337)	-1.8623 (3.4726)	0.6331 (0.4009)	0.5782 (0.2568)	0.5087 (0.1321)	-0.7252 (0.5271)	-0.7039 (3.0891)	0.7
-1.7148 (2.9456)	-1.7407 (3.0361)	-1.7715 (3.1458)	0.6084 (0.3702)	0.5545 (0.2362)	0.5267 (0.1212)	-0.7382 (0.5018)	-0.7284 (3.3964)	0.8
-1.8433 (3.4020)	-1.8679 (3.4943)	-1.8972 (3.6056)	0.6665 (0.4443)	0.6072 (0.2832)	0.5550 (0.1455)	-0.7826 (0.6128)	-0.7583 (4.6740)	0.9
<i>The average, (mean square error) of the estimators of parameter β</i>								
0.5549 (0.0211)	0.5571 (0.0204)	0.5598 (0.0197)	0.5558 (0.0208)	0.5626 (0.0188)	0.5740 (0.0173)	0.5876 (0.0129)	0.5680 (0.0882)	0.7
0.6494 (0.0227)	0.6520 (0.0219)	0.6551 (0.0210)	0.6546 (0.0212)	0.6629 (0.0188)	0.6744 (0.0170)	0.6919 (0.0118)	0.6696 (0.1005)	0.8
0.7455 (0.0239)	0.7483 (0.0231)	0.7518 (0.0220)	0.7558 (0.0208)	0.7659 (0.018)	0.7794 (0.0158)	0.8019 (0.0098)	0.7743 (0.1085)	0.9
<i>The average, (mean square error) of the estimators of reliability function $R(t)$</i>								
$R(t=2)=0.0543$								0.7
0.0465 (0.0001)	0.0464 (0.0001)	0.0463 (0.0001)	0.0072 (0.0022)	0.0073 (0.0022)	0.0078 (0.0022)	0.0456 (0.0001)	0.0445 (0.0005)	
$R(t=2)=0.0759$								0.8
0.0679 (0.0001)	0.0676 (0.0001)	0.0674 (0.0001)	0.0151 (0.0037)	0.0155 (0.0036)	0.0168 (0.0035)	0.0659 (0.0001)	0.0639 (0.0009)	
$R(t=2)=0.0978$								0.9
0.0906 (0.0001)	0.0901 (0.0001)	0.0897 (0.0001)	0.0239 (0.0055)	0.0247 (0.0053)	0.0270 (0.0050)	0.0873 (0.0001)	0.0847 (0.0013)	
<i>The average, (mean square error) of the estimators of hazard rate function $H(t)$</i>								
$H(t=2)=1.3510$								0.7
1.6284 (0.0772)	1.6289 (0.0776)	1.6309 (0.0788)	1.7816 (0.1891)	1.9736 (0.3968)	2.3307 (0.9855)	(2×10^{14}) (4×10^{28})	(2×10^{14}) (3×10^{29})	
$H(t=2)=1.1552$								0.8
1.3710 (0.0468)	1.3728 (0.0477)	1.3765 (0.0493)	1.4895 (0.1139)	1.6413 (0.2417)	1.9229 (0.6043)	(3×10^{14}) (10×10^{28})	(3×10^{14}) (1×10^{30})	
$H(t=2)=1.0025$								0.9
1.1717 (0.0289)	1.1745 (0.0299)	1.1794 (0.0317)	1.2622 (0.0691)	1.3841 (0.1497)	1.6109 (0.3815)	(1×10^{14}) (1×10^{28})	(1×10^{14}) (1×10^{29})	

From the simulation studies we noted that:

1. In general, the Bayesian estimators have mean square error less than that of the MLE.
2. Increasing the sample size leads to decrease mean square error and increase the accuracy of estimators.
3. The estimate of μ under general entropy loss function is the best especially at the value $q = 5$, and it followed by the MLE. Also by decreasing the value of the parameter β , the accuracy of estimates increases and mean square error decreases.
4. For the parameter β , the estimate under squared error loss function is the best, and it followed by the general entropy loss function especially when the value $q = 5$.
5. The estimate of the reliability function $R(t)$ under linex loss function is the best, and it followed by the squared error loss function.
6. For the hazard rate function $H(t)$, the estimate under linex loss function is the best, and it followed by the general entropy loss function.

5. Concluding remarks

Bayesian estimators of the two parameters, reliability, and hazard rate functions for the Logistic distribution using Lindley's approximation, based on progressively type-II censoring samples are obtained. We assumed non-informative prior distributions for the parameters. Computer simulation study is performed, and show that increasing the sample size leads to decrease mean square error and increase the accuracy of estimators. The simulation also stresses the importance of linex and general Entropy loss functions are applicable in the case studied.

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