On Five-Parameter Lomax Distribution: Properties and Applications

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Abstract

A five-parameter continuous model, called the beta exponentiated Lomax distribution, is defined and studied. The model has as special sub-models some important lifetime distributions discussed in the literature, such as the logistic, Lomax, exponentiated Lomax, beta Lomax distributions, among several others. We derive the ordinary and incomplete moments, generating and quantile functions, mean deviations, Bonferroni, Lorenz and Zenga curves, mean residual life, mean waiting time and Rényi of entropy. The method of maximum likelihood is proposed for estimating the model parameters. We obtain the observed information matrix. Three real data sets demonstrate that the new distribution can provide a better fit than other classical lifetime models.

Keywords: Beta distribution; Lomax distributions; Maximum likelihood estimation.

1. Introduction

The Lomax (or Pareto II) distribution has wide applications in many fields such as income and wealth inequality, medical and biological sciences, engineering, size of cities actuarial science, lifetime and reliability modeling. In the lifetime context, the Lomax model belongs to the family of decreasing failure rate (see Chahkandi and Ganjali, 2009) and arises as a limiting distribution of residual lifetimes at great age (see Balkema and de Hann, 1974). For more information about the Lomax distribution and Pareto family are given in Arnold (1983) and Johnson et al. (1994). Various generalizations of Lomax distribution have been studied, the exponentiated Lomax, discussed by Abdul-Moniem and Abdel-Hameed (2012), Marshall-Olkin extended Lomax defined by Ghitany et al. (2007), McDonald Lomax investigated by Lemonte and Cordeiro (2013), gamma Lomax introduced by Cordeiro et al. (2015), the Weibull Lomax distribution studied by Tahir et al. (2015) and recently the transmuted Weibull Lomax distribution given by Afify et al. (2015).

The random variable X with exponentiated Lomax (EL) distribution has the cumulative distribution function (cdf) given by

$$G(x;\lambda,\theta,\beta)=\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}, \tag{1}$$

for $\lambda>0,\ \theta>0,\ \beta>0$ and $x\geq 0$. The probability density function (pdf) corresponding to (1) takes the form

$$g(x;\lambda,\theta,\beta)=\beta\,\theta\,\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\beta-1}, \tag{2}$$

2. The Beta Exponentiated Lomax Distribution

Let $G(x)$ be the cdf of any random variable X. The cdf of a generalized class of distributions defined by Eugene et al. (2002) is given by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} v^{a-1}(1-v)^{b-1} dv, \tag{3}$$

where $a > 0, b > 0$ are the shape parameters, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y v^{a-1}(1-v)^{b-1} dv$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(.)$ is the gamma function. The corresponding pdf for (3) is given by

$$f(x) = \frac{1}{B(a,b)} g(x)\left[G(x)\right]^{a-1}\left[1-G(x)\right]^{b-1}, \tag{4}$$

where $g(x) = \partial G(x)/\partial x$ is the baseline density function. Replacing (1) in (3), we obtain a new distribution, called beta exponentiated Lomax (BEL), with cdf given by

$$F\left(x,\zeta\right) = I_w(a,b) = \frac{1}{B(a,b)} \int_0^w v^{a-1}(1-v)^{b-1} dv, \tag{5}$$

for $a > 0$ and $b > 0$. Here $w = \left(1-(1+\lambda x)^{-\theta}\right)^{\beta}$ and $\zeta = (a,b,\lambda,\theta,\beta)$ is the vector of the model parameters. Equation (5) can be expressed as follows

$$F(x;\zeta) = \frac{\left(1-(1+\lambda x)^{-\theta}\right)^{a\beta}}{a\,B(a,b)}\left[{}_2F_1\left(a, 1-b\,; a+1; \left(1-(1+\lambda x)^{-\theta}\right)^{\beta}\right)\right],$$

where

$${}_2F_1\left(a,b\,;c\,;z\right) = \frac{1}{B(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-t\,z)^a} dt,$$

is the well known hypergeometric function (Gradshteyn and Ryzhik, 2007).

The pdf corresponding to (5) is given by

$$f(x;\xi) = \frac{\beta\theta\lambda}{B(a,b)}(1+\lambda x)^{-(\theta+1)}\left(1-(1+\lambda x)^{-\theta}\right)^{a\beta-1}\left[1-\left(1-(1+\lambda x)^{-\theta}\right)^{\beta}\right]^{b-1}. \tag{6}$$

In **Fig. 1**, we Plot of the BEL density function for different values of $(a,b,\beta,\theta,\lambda)$.

For a lifetime random variable t, the survival function S(t), hazard rate function h(t), reversed hazard rate function r(t) and the cumulative hazard rate function H(t) of BEL distribution are given by

$$S(t) = 1 - F(t) = 1 - I_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b),$$

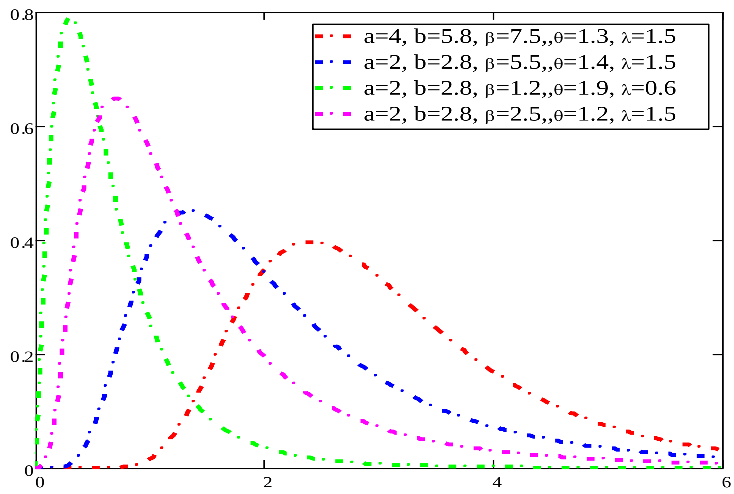

Fig. 1. The pdf of the BEL for different values of the parameters

$$h(t)=\frac{f(t)}{S(t)}=\frac{\beta\theta\lambda(1+\lambda t)^{-(\theta+1)}\left(1-(1+\lambda t)^{-\theta}\right)^{a\beta-1}\left[1-\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}\right]^{b-1}}{\left[B(a,b)-B_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b)\right]}, \tag{7}$$

$$r(t)=\frac{f(t)}{F(t)}=\frac{\beta\theta\lambda(1+\lambda t)^{-(\theta+1)}\left(1-(1+\lambda t)^{-\theta}\right)^{a\beta-1}\left[1-\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}\right]^{b-1}}{B_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b)} \tag{8}$$

and

$$H(t)=-\ell n\,S(t)=-\ell n\left[1-I_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b)\right].$$

Plots of the HRF for different values of $(a,b,\beta,\theta,\lambda)$ are given in **Fig. 2**.

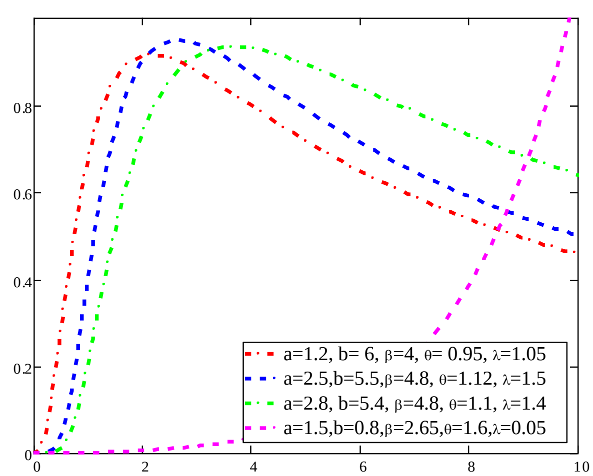

Fig. 2. The HRF of the BEL for different values of the parameters

2.1 Sub-models

The following distributions are special of the BEL distribution:

1. When $\beta=1,$ equation (6) reduces to the beta Lomax (BL) distribution.
2. Setting $\lambda=1,$ we obtain the beta exponentiated standard Lomax distribution (BESL) distribution.
3. If $b=\beta=1,$ the density (6) corresponds to the exponentiated Lomax distribution (EL) distribution.
4. When $a=b=1,$ we obtain the exponentiated Lomax distribution (EL) distribution.
5. If $a=b=\lambda=1,$ the BEL gives the exponentiated standard Lomax (ESL) distribution.
6. Equation (6) becomes the Lomax distribution for the choice $a=b=\beta=1$
7. Setting $a=b=\beta=\lambda=1,$ the density (6) yields the standard Lomax (SL) distribution.

3. Some Statistical Properties

We give a mathematical treatment of the new distribution including expansions for the density function, moments, incomplete moments, quantile function, mean deviations, Bonferroni, Lorenz and Zenga curves, mean residual life, mean waiting time and Rényi entropy.

3.1 Expansions for the Distribution and Density Functions

Equations (5) and (6) are straightforward to compute using any statistical software. However, we obtain expansions for $F(x)$ and $f(x)$ in terms of an infinite (or finite) weighted sums of cdf's and pdf's of random variables having Lomax distributions, respectively. For any positive real number b and for $|z|<1$, a generalized binomial expansion holds

$$\left(1-z\right)^{b-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(b)}{j!\Gamma(b-j)}z^{j}. \tag{9}$$

Therefore, the cdf of BEL can be expanded to obtain

$$F\left(x,\zeta\right)=I_{w}(a,b)=\frac{1}{B(a,b)}\int_{0}^{w}v^{a-1}(1-v)^{b-1}dv,$$

$$\begin{aligned}F(x,\zeta)&=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_{0}^{w}v^{a+j-1}\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(b)}{j!\Gamma(b-j)}\,dv\\&=\sum_{j=0}^{\infty}p_{j}G(x;\lambda,\theta,\beta(a+j)),\end{aligned} \tag{10}$$

where

$$p_j = \frac{(-1)^j \Gamma(a+b)}{\Gamma(a)\, j!\, \Gamma(b-j)(a+j)},$$

and $G(x;\lambda,\theta,\beta(a+j))$ denotes the cdf of EL with parameters λ, θ and $\beta(a+j)$.

Similarly, we can write the pdf (6) as

$$\begin{aligned} f(x;\zeta) &= \sum_{j=0}^{\infty} \frac{(-1)^j \beta\theta\lambda\Gamma(a+b)}{\Gamma(a)\Gamma(b-j)} (1+\lambda x)^{-(\theta+1)} \left(1-(1+\lambda x)^{-\theta}\right)^{\beta(a+j)-1} \\ &= \sum_{j=0}^{\infty} p_j\, H(x;\lambda,\theta,\beta(a+j)), \end{aligned} \tag{11}$$

where $H(x;\lambda,\theta,\beta(a+j))$ denotes the EL density function with parameters λ, θ and $\beta(a+j)$.

Again, by using binomial expansion in equation (11), we obtain

$$\begin{aligned} f(x,\zeta) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\beta\lambda\theta(-1)^{j+i}\Gamma(a+b)\Gamma(\beta(a+j))}{i!\,\Gamma(a)\, j!\,\Gamma(b-j)\Gamma(\beta(a+j)-i)} (1+\lambda x)^{-\theta(i+1)-1} \\ &= \sum_{i=0}^{\infty} q_i\, h(x;\lambda,\theta(i+1)), \end{aligned} \tag{12}$$

where

$$q_i = \sum_{j=0}^{\infty} \frac{\beta(-1)^{j+i}\Gamma(a+b)\Gamma(\beta(a+j))}{i!\,\Gamma(a)\, j!(i+1)\Gamma(b-j)\Gamma(\beta(a+j)-i)},$$

and $h(x;\lambda,\theta(i+1))$ denotes the Lomax density with parameters λ and $\theta(i+1)$. If b is an integer, then the summation in equations (10), (11) and (12) stops at $b-1$.Thus, the BEL density function can be expressed as an infinite linear combination of Lomax densities. Thus, some of its mathematical properties can be obtained directly from those properties of the Lomax distribution.

3.2 Moments and Moment Generating Function

As with any other distribution, many of the interesting characteristics and features of the BEL distribution can be studied through the moments. If we assume that Y is a Lomax distributed random variable, with parameters λ and θ, then the rth moment of Y is given

$$E(Y^r) = (\theta/\lambda^r)\, B\left(r+1,\, \theta-r\right), \qquad \theta > r.$$

Let X be a random variable having the BEL distribution (6). Using equation (12), it is easy to obtain the rth moment of X as

$$E(X^r) = \sum_{i=0}^{\infty} q_i\, (\theta(i+1)/\lambda^r)\, B\left(r+1,\, \theta(i+1)-r\right), \qquad \theta > r. \tag{13}$$

The mean, variance, Skewness and Kurtosis can be obtained from (13). If $b>0$ is integer and $\theta(i+1) > r$, the sum stops at $b-1$.

The moment generating function (mgf), say $M(t) = E[\exp(tX)]$ of BEL is given by

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$$

where $E(X^k)$ follows from equation (13).

3.3 Quantile Function

Let $Q_{a,b}(u)$ be the beta quantile function with parameters a and b. The quantile function of the BEL distribution, say $x = Q(u)$, can be easily obtained as

$$x = Q(u) = \lambda^{-1}\left[\left[1-\left(Q_{a,b}(u)\right)^{1/\beta}\right]^{-(1/\theta)} - 1\right], \qquad u \in (0,1). \tag{14}$$

This scheme is useful to generate BEL random variates because of the existence of fast generators for beta random variables in most statistical packages, i.e. if V is a beta random variable with parameters a and b, then

$$X = \lambda^{-1}\left[\left[1-V^{1/\beta}\right]^{-(1/\theta)} - 1\right],$$

follows the BEL distribution. From (14) we conclude that the median m of X is $m = Q(1/2)$.

The Bowley skewness (SK) measure and Moors kurtosis (KR) (based on octiles) of the BEL distribution can be calculated using the formulae given below

$$SK = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

and

$$KR = \frac{[Q(7/8) - Q(5/8)] + [Q(3/8) - Q(1/8)]}{[Q(6/8) - Q(2/8)]}.$$

3.4 Incomplete Moments

If Y is a random variable with BXII distribution with parameters λ and θ, the rth incomplete moment of Y, for $\theta > r$, is given by

$$m_r(z) = \int_0^z y^r g(y;\lambda,\theta)\, dy = (\theta/\lambda^r)\, B_{\lambda z}\left(r+1,\ \theta - r\right), \qquad \theta > r.$$

From this equation, we note that $M_s(z) \to E(Y^s)$ when $z \to \infty$, whenever $\theta > r$. Let X be a random variable having the BEL distribution (6). The rth incomplete moment of X is then equal to

$$m_r(z) = \sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda^r)\, B_{\lambda z}\left(r+1,\ \theta(i+1) - r\right), \qquad \theta(i+1) > r. \tag{15}$$

3.5 Mean Deviations

The mean deviations about the mean and the median can be used as measures of spread in a population. Let $\mu = E(X)$ and θ be the mean and the median of the BEBXII distribution respectively. The mean deviations about the mean and about the median of X can be calculated as

$$D(\mu) = E\left|X-\mu\right| = \int_d^\infty |x-\mu|\, f(x)\, dx = 2\mu F(\mu) - 2m_1(\mu)$$

and

$$D(\theta) = E\left|X-\theta\right| = \int_d^\infty |x-\theta|\, f(x)\, dx = \mu - 2m_1(\theta),$$

respectively, where $m_1(\mu)$ denotes the first incomplete moment and $F(\mu)$ follows from (5).

3.6 Mean Residual Life and Mean Waiting Time

The mean residual life function (MRL) at a given time t measures the expected remaining lifetime of an individual of age t. It is denoted by $m(t)$. The MRL or life expectancy of BEL is defined as

$$m(t) = \frac{1}{S(t)}\left[E(t) - \int_0^t t\, f(t)\, dt\right] - t,$$

$$= \frac{\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\left[B\left(2,\ \theta(i+1)-1\right) - B_{\lambda t}\left(2,\ \theta(i+1)-1\right)\right]}{1 - I_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b)} - t, \qquad \theta > 1,$$

where

$$\int_0^t t\, f(t)\, dt = \sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\, B_{\lambda t}\left(2,\ \theta(i+1)-1\right).$$

The mean waiting time (MWT) of an item failed in a interval [0, t] for BEL is defined as

$$\bar{\mu}(t,\theta) = t - \frac{1}{F(t)}\int_0^t t\, f(t)\, dt$$

$$= t - \frac{\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\, B_{\lambda t}\left(2,\ \theta(i+1)-1\right)}{I_{\left(1-(1+\lambda t)^{-\theta}\right)^{\beta}}(a,b)}, \qquad \theta > 1.$$

3.7 Lorenz, Bonferroni and Zenga Curves

Lorenz and Bonferroni curves have been applied in many fields such as economics, reliability, demography, insurance and medicine, (see Kleiber and Kotz, (2003) for additional details). Zenga curve was presented by Zenga (2007). The Lorenz $L_F(x)$, Bonferroni $B(F(x))$ and Zenga $A(x)$ curves are defined by Oluyede and Rajasooriya (2013) as the following

$$L_F(x)=\int_0^x t\,f(t)\,dt \Big/ E(X),\quad B(F(x))=\int_0^x t\,f(t)\,dt \Big/ F(x)E(X)=L_F(x)/F(x)\quad \text{and}\quad A(x)=1-\left[\overset{-}{\mathrm{M}}(x)\Big/\overset{+}{\mathrm{M}}(x)\right]$$

respectively, where

$$\overset{-}{\mathrm{M}}(x)=\int_0^x t\,f(t)\,dt \Big/ F(x)\quad \text{and}\quad \overset{+}{\mathrm{M}}(x)=\int_x^\infty t\,f(t)\,dt \Big/ 1-F(x)$$

are the lower and upper means respectively. For the BEL distribution, these quantities are derived below

1. Lorenz curve:

$$L_{FG}(x;\zeta)=\frac{\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\,B_{\lambda x}\left(2,\ \theta(i+1)-1\right)}{\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\,B\left(2,\ \theta(i+1)-1\right),}.$$

2. Bonferroni curve:

$$B(F_G(x;\zeta))=\frac{\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\,B_{\lambda x}\left(2,\ \theta(i+1)-1\right)}{\sum_{j=0}^{\infty} p_j G(x;\lambda,\theta,\beta(a+j))\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\,B\left(2,\ \theta(i+1)-1\right)}.$$

3. Zenga curve:

$$A(x;\zeta)=1-\frac{\left[1-F(x)\right]\left[\int_0^x t\,f(t)\,dt\right]}{F(x)\int_x^\infty t\,f(t)\,dt},$$

$$=1-\frac{\left[1-\sum_{j=0}^{\infty} p_j G(x;\lambda,\theta,\beta(a+j))\right]\left[\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\,B_{\lambda x}\left(2,\ \theta(i+1)-1\right)\right]}{\left[\sum_{j=0}^{\infty} p_j G(x;\lambda,\theta,\beta(a+j))\right]\left[\sum_{i=0}^{\infty} q_i\ (\theta(i+1)/\lambda)\left[B\left(2,\ \theta(i+1)-1\right)-B_{\lambda t}\left(2,\ \theta(i+1)-1\right)\right]\right]}.$$

3.8 Rényi Entropy

The entropy of a random variable X is a measure of uncertainty variation. The Rényi entropy is defined as

$$I_R(\delta)=\frac{1}{1-\delta}\left[\log I(\delta)\right],$$

where $I(\delta)=\int f^{\delta}(x)\,dx,\quad \delta>0$ and $\delta\neq 1$. Using equation (5) we obtain

$$I(\delta)=\frac{\beta^{\delta}\,\theta^{\delta}\,\lambda^{\delta}}{\beta^{\delta}(a,b)}\int_0^\infty (1+\lambda x)^{-\delta(\theta+1)}\left(1-(1+\lambda x)^{-\theta}\right)^{\delta(a\beta-1)}\left[1-\left(1-(1+\lambda x)^{-\theta}\right)^{\beta}\right]^{\delta(b-1)}dx.$$

Based on the binomial expansion to the last factor in the above integrand yields

$$I(\delta)=\frac{\beta^{\delta}\,\theta^{\delta}\,\lambda^{\delta}}{\beta^{\delta}(a,b)}\sum_{i=0}^{\infty}\binom{\delta(b-1)}{i}(-1)^{i}\int_{0}^{\infty}(1+\lambda x)^{-\delta(\theta+1)}\left(1-(1+\lambda x)^{-\theta}\right)^{\delta(a\beta-1)+i\beta}dx.$$

Again, using the binomial expansion to the last factor, we obtain

$$I(\delta)=\frac{\beta^{\delta}\,\theta^{\delta}\,\lambda^{\delta}}{\beta^{\delta}(a,b)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\delta(b-1)}{i}\binom{\delta(a\beta-1)+i\beta}{j}(-1)^{i+j}\int_{0}^{\infty}(1+\lambda x)^{-\delta(\theta+1)-j\theta}\,dx.$$

Using integration in above expression and simplifying,

$$I(\delta)=\frac{\beta^{\delta}\,\theta^{\delta}\,\lambda^{\delta-1}}{\beta^{\delta}(a,b)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\delta(b-1)}{i}\binom{\delta(a\beta-1)+i\beta}{j}(-1)^{i+j}\left[\delta(\theta+1)+j\theta-1\right]^{-1}.$$

Hence, the Rényi entropy reduces to

$$I_R(\delta)=\left(\frac{1}{\delta-1}\right)\left[\delta\log\left(\frac{\beta\theta}{B(a,b)}\right)+\log\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\delta(b-1)}{i}\binom{\delta(a\beta-1)+i\beta}{j}(-1)^{i+j}\left[\delta(\theta+1)+j\theta-1\right]^{-1}\right]+\log(\lambda).$$

4. Estimation of Parameters

The maximum likelihood estimation (MLE) is one of the most widely used estimation method for finding the unknown parameters. Let $X_1, X_2, \ldots, X_n$ be an independent random sample from BEBXII. The total log-likelihood is given by

$$\ell=n\ell n(\beta)+n\ell n(\theta)+n\ell n(\lambda)-n\ell n\,B(a,b)-(\theta+1)\sum_{i=1}^{n}\ell n(u_i)+(a\beta-1)\sum_{i=1}^{n}\ell n(z_i)$$
$$+(b-1)\sum_{i=1}^{n}\ell n(1-z_i^{\beta}), \qquad (16)$$

where $u_i=(1+\lambda x_i)$ and $z_i=(1-u_i^{-\theta})$.

The score vector $\nabla\ell=(\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\beta})$ has components

$$\frac{\partial\ell}{\partial a}=n\left(\psi(a+b)-\psi(a)\right)+\beta\sum_{i=1}^{n}\ell n(z_i),$$

$$\frac{\partial\ell}{\partial b}=n\left(\psi(a+b)-\psi(b)\right)+\sum_{i=1}^{n}\ell n(1-z_i^{\beta}),$$

$$\frac{\partial\ell}{\partial\lambda}=n/\lambda+\theta(a\beta-1)\sum_{i=1}^{n}z_i^{-1}u_i^{-(\theta+1)}x_i-\beta\theta(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-(\theta+1)}x_i-(\theta+1)\sum_{i=1}^{n}u_i^{-1}x_i,$$

$$\frac{\partial\ell}{\partial\theta}=n/\theta-\sum_{i=1}^{n}\ell n(u_i)+(a\beta-1)\sum_{i=1}^{n}z_i^{-1}u_i^{-\theta}\ell n(u_i)-\beta(b-1)\sum_{i=1}^{n}\left(1-z_i^{\beta}\right)^{-1}z_i^{\beta-1}u_i^{-\theta}\ell n(u_i)$$

$$\frac{\partial \ell}{\partial \beta} = n/\beta + a\sum_{i=1}^{n} \ell n(z_i) - (b-1)\sum_{i=1}^{n}\left(1 - z_i^{\beta}\right)^{-1} z_i^{\beta} \ell n(z_i)$$

where $\psi(p)$ is the digamma function which is the derivative of log $\Gamma(.)$. The maximum likelihood estimates (MLEs) of the unknown five parameters can be obtained by solving the system of nonlinear equations $\nabla \ell = 0$, iteratively.

For interval estimation of $(a,b,\lambda,\theta,\beta))$ and hypothesis tests on these parameters, we obtain the observed information matrix since its expectation requires numerical integration. The 5×5 observed information matrix $J(\alpha)$ is

$$J(\alpha) = \begin{bmatrix} J_{aa} & J_{ab} & J_{a\lambda} & J_{a\theta} & J_{a\beta} \\ J_{ba} & J_{bb} & J_{b\lambda} & J_{b\theta} & J_{b\beta} \\ J_{\lambda a} & J_{\lambda b} & J_{\lambda\lambda} & J_{\lambda\theta} & J_{\lambda\beta} \\ J_{\theta a} & J_{\theta b} & J_{\theta\lambda} & J_{\theta\theta} & J_{\theta\beta} \\ J_{\beta a} & J_{\beta b} & J_{\beta\lambda} & J_{\beta\theta} & J_{\beta\beta} \end{bmatrix},$$

whose elements are given in Appendix.

5. Applications

In this section we provide three applications of the BEL distribution to three real data sets. The first data set, strength data, which were originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10 mm with sample size (n = 63). This data set consists of observations: 1.901, *2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.* This data set is previously studied by Afify et al. (2015) to fit the exponentiated transmuted generalized Rayleigh distribution.

As a second application, we analyze a real data set on the active repair times (hours) for an airborne communication transceiver. This data set was analyzed by Jorgensen (1982). The data are as follows: *0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50*. Recently, Lemont et al. (2013) studied these data using the exponentiated Kumaraswamy distribution. Based on the third application, we use the lifetime data set given by Gross and Clark (1975). Their data set represents the relief times of twenty patients receiving an analgesic. The data are as follows: *1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0* . Recently, this data set is previously studied by Rodrigues et al. (2014) to fit the beta exponentiated Lindley distribution.

We use these three data sets to compare the fit of the BEL distribution (and their sub-models, BL and EL) with four models: McDonald Lomax (McL) (Lemonte and Cordeiro, 2013), gamma Lomax (GL) (Cordeiro et al., 2015), Weibull Lomax (WL) (Tahir et al., 2015) and the transmuted Weibull Lomax (TWL) (Afify et al., 2015).

Tables 1, 2 and 3 list the maximum likelihood estimates MLEs (the corresponding standard errors in parentheses) of the parameters of all the models for the three data sets respectively.

Table 1: MLEs (standard errors in parentheses) for BEL, BL, EL, McL, TWL, WL and GL models and the statistics -2ℓ, W^* and A^*; first data set

Model	Estimates					Statistics		
						-2ℓ	W^*	A^*
BEL$(a,b,\beta,\theta,\lambda)$	29.53059 (101.309)	44.79472 (106.061)	20.966 (96.726)	0.78519 (0.721)	17.98555 (79.283)	**112.604**	**0.05645**	**0.31003**
BL(a,b,θ,λ)	38.16636 (31.055)	23.92507 (135.907)	2.80895 (13.437)	0.13505 (0.321)	-	113.106	0.05754	0.33462
EL(θ,λ,β)	397.3952 (640.182)	37.79907 (102.78)	0.06217 (0.203)	-	-	113.515	0.07394	0.39048
McL$(a,b,\beta,\theta,\lambda)$	37.12441 (37.783)	26.14064 (236.492)	3.68277 (10.640)	1.89877 (14.485)	2.85382 (5.007)	113.018	0.05712	0.32925
TWL$(a,b,\beta,\theta,\lambda)$	0.90616 (5.337)	9.53727 (5.692)	0.46199 (0.961)	0.32117 (0.276)	0.73397 (0.288)	119.546	0.1058	0.72029
WL(a,b,β,θ)	5.0955 (27.064)	10.02316 (6.5904)	0.32753 (0.784)	0.25575 (0.212)	-	121.828	0.11789	0.81509
GL(a,θ,β)	50.15703 (36.5936)	67.73728 (29.078)	2.76071 (4.312)	-	-	112.922	0.05965	0.33634

Table 2: MLEs (standard errors in parentheses) for BEL, BL, EL, McL, TWL, WL and GL models and the statistics -2ℓ, W^* and A^*; second data set.

Model	Estimates					Statistics		
						-2ℓ	W^*	A^*
BEL$(a,b,\beta,\theta,\lambda)$	11.60343 (39.574)	4.74111 (11.378)	1.78971 (5.085)	0.45684 (0.736)	22.55858 (80.441)	**179.286**	**0.05397**	**0.36446**
BL(a,b,θ,λ)	10.7189 (14.179)	1.26299 (4.261)	1.12572 (2.822)	3.89665 (8.224)	-	179.515	0.06206	0.40133
EL(β,θ,λ)	2.19098 (0.6066)	3.08118 (1.787)	0.22458 (0.249)	-	-	183.563	0.06845	0.51556
McL$(a,b,\beta,\theta,\lambda)$	3.05238 (2.631)	5.98506 (9.286)	4.37967 (4.746)	0.45115 (0.932)	0.09045 (0.813)	182.733	0.07782	0.53932
TWL$(a,b,\beta,\theta,\lambda)$	38.90099 (125.562)	2.66124 (1.117)	0.28393 (0.457)	0.07811 (0.076)	0.53887 (0.557)	182.397	0.08218	0.56988
WL(a,b,β,θ)	29.23558 (74.284)	2.37142 (0.9158)	0.3235 (0.479)	0.08792 (0.075)	-	183.171	0.08853	0.6263
GL(a,θ,β)	14.02425 (19.8392)	3.91 (2.556)	0.06753 (0.198)	-	-	179.45	0.05422	0.37053

Table 3: MLEs (standard errors in parentheses) for BEL, BL, EL, McL, TWL, WL and GL models and the statistics -2ℓ, W^* and A^*; third data set

Model	Estimates					Statistics		
						-2ℓ	W^*	A^*
BEL$(a,b,\beta,\theta,\lambda)$	12.57495 (22.35)	2.21626 (6.303)	8.95537 (15.436)	3.45605 (8.635)	1.31206 (3.937)	**31.589**	**0.0413**	**0.23478**
BL(a,b,θ,λ)	41.07035 (41.274)	1.92859 (2.348)	5.77401 (9.086)	0.42885 (0.734)	-	32.219	0.04951	0.28878
EL(θ,λ,β)	59.45778 (65.646)	14.36113 (20.4)	0.20591 (0.386)	-	-	31.7096	0.04318	0.24788
McL$(a,b,\beta,\theta,\lambda)$	14.17225 (10.705)	16.48026 (37.316)	8.40821 (14.158)	2.75535 (6.654)	0.60938 (1.992)	34.561	0.08965	0.52724
TWL$(a,b,\beta,\theta,\lambda)$	8.61876 (42.832)	6.2149 (4.501)	0.24791 (0.666)	0.22551 (0.202)	0.69656 (0.338)	37.804	0.13191	0.79539
WL(a,b,β,θ)	14.7394 (64.67)	5.58544 (3.8398)	0.26331 (0.673)	0.21908 (0.184)	-	39.261	0.14848	0.90647
GL(a,θ,β)	26.50612 (24.4554)	25.31335 (8.866)	0.9907 (1.642)	-	-	33.22	0.06657	0.39107

The statistics: -2ℓ (where ℓ denotes the log-likelihood function evaluated at the maximum likelihood estimates), the Anderson–Darling (A^*) and Cramér-von Mises (W^*) are reported in Tables 1, 2 and 3. In general, the distribution which has the smaller values of these statistics is the better the fit to the data. The results show that the BEL distribution provides a significantly better fit than the other six models. All the computations were done using the MATH- CAD PROGRAM.

6. Concluding remarks

In this paper, we proposed a new distribution, named the beta exponentiated Lomax distribution which extends the Lomax distribution. Several properties of the new distribution were investigated, including ordinary and incomplete moments, mean deviations, Rényi entropy, and reliability. The model parameters are estimated by maximum likelihood and the information matrix is derived. Three applications of the beta exponentiated Lomax distribution to real data show that the new distribution can be used quite effectively to provide better fits than the exponentiated Lomax (Abdul-Moniem and Abdel-Hameed, 2012), beta Lomax and McDonald Lomax (Lemonte and Cordeiro, 2013), Weibull Lomax (Tahir et al., 2015), gamma Lomax (Cordeiro et al., 2015) and recently, the transmuted Weibull Lomax (Afify et al., 2015). We hope that the proposed model may attract wider applications in many areas such as engineering, survival analysis, hydrology, economics, and so on.

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Appendix

The elements of the observed information matrix $J(\alpha)$ for the parameters $(a,b,\lambda,\theta,\beta)$ are

$$J_{aa}=-n\left(\psi'(a+b)-\psi'(a)\right),$$

$$J_{ab}=-n\left(\psi'(a+b)\right),$$

$$J_{a\lambda}=-\beta\theta\sum_{i=1}^{n} z_i^{-1}u_i^{-\theta-1}x_i\,,$$

$$J_{a\theta}=-\beta\sum_{i=1}^{n} z_i^{-1}u_i^{-\theta}\ell(u_i),$$

$$J_{a\beta}=-\sum_{i=1}^{n}\ell n(z_i),$$

$$J_{bb}=-n\left(\psi'(a+b)-\psi'(b)\right),$$

$$J_{b\lambda}=\beta\theta\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta-1}x_i,$$

$$J_{b\theta}=\beta\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta}\ell n(u_i),$$

$$J_{b\beta}=\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta}\,\ell n(z_i),$$

$$J_{\lambda\lambda}=n/\lambda^2-(\theta+1)\sum_{i=1}^{n}u_i^{-2}x_i^2+\theta(a\beta-1)\sum_{i=1}^{n}z_i^{-1}u_i^{-(\theta+2)}x_i^2\left(\theta z_i^{-1}u_i^{-\theta}+(\theta+1)\right)+\beta\theta(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-(\theta+1)}x_i^2,$$

$$J_{\lambda\theta}=\beta(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-(\theta+1)}x_i\left[-\theta\ell n(u_i)+1+\beta\theta(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta}\ell n(u_i)+\theta z_i^{-1}u_i^{-\theta}\ell n(u_i)\right]$$
$$+\sum_{i=1}^{n}u_i^{-1}x_i-(a\beta-1)\sum_{i=1}^{n}z_i^{-1}u_i^{-(\theta+1)}x_i\left[-\theta\,\ell n(u_i)+1+\theta\,u_i^{-\theta}\,\ell n(u_i)\right],$$

$$J_{\lambda\beta}=\theta(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-(\theta+1)}\,x_i\left[\beta(1-z_i^{\beta})^{-1}z_i^{\beta}\ell n(z_i)+\beta\ell n(z_i)+1\right]-\theta a\sum_{i=1}^{n}z_i^{-1}u_i^{-(\theta+1)}\,x_i\,,$$

$$J_{\theta\theta}=\beta(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta}\,\ell n^2(u_i)\left[\beta(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta}+(\beta-1)z_i^{-1}u_i^{-\theta}-1\right]$$
$$+n/\theta^2+(a\beta-1)\sum_{i=1}^{n}z_i^{-1}u_i^{-\theta}\,\ell n^2(u_i)\left[1+u_i^{-\theta}z_i^{-1}\right],$$

$$J_{\theta\beta}=(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta-1}u_i^{-\theta}\,\ell n(u_i)\left[\beta(1-z_i^{\beta})^{-1}z_i^{\beta}\ell n(z_i)+\beta\ell n(z_i)+1\right]-a\sum_{i=1}^{n}z_i^{-1}u_i^{-\theta}\,\ell n(u_i),$$

$$J_{\beta\beta}=n/\beta^2+(b-1)\sum_{i=1}^{n}(1-z_i^{\beta})^{-1}z_i^{\beta}\ell n^2(z_i)\left[(1-z_i^{\beta})^{-1}z_i^{\beta}+1\right].$$