A Generalized Gamma-Weibull Distribution: Model, Properties and Applications

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Abstract

We prepare a new method to generate family of distributions. Then, a family of univariate distributions generated by the Gamma random variable is defined. The generalized gamma-Weibull (GGW) distribution is studied as a special case of this family. Certain mathematical properties of moments are provided. To estimate the model parameters, the maximum likelihood estimators and the asymptotic distribution of the estimators are discussed. Certain characterizations of GGW distribution are presented. Finally, the usefulness of the new distribution, as well as its effectiveness in comparison with other distributions, are shown via an application of a real data set.

Keywords: Gamma-generated distribution, Generalized gamma-Weibull distribution, Weibull distribution, Hazard function, Maximum likelihood estimation, Characterizations.

1. Introduction

Recently, some attempts have been made to define new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in modelling data in practice. A common feature of these generalized distributions is that they have more parameters. Zografos and Balakrishnan (2009) and Torabi and Montazeri (2010) defined the Type I and II family of "Gamma-generated" distributions, respectively. Alzaatre at al. (2013) developed a new method to generate family of distributions and called it the $T-X$ family of distributions. Barreto-Souza and Simas (2013) defined a class of distributions given by

$$F(x)=\begin{cases}\dfrac{1-e^{\lambda G(x)}}{1-e^{\lambda}} & \lambda\neq 0\\ G(x) & \lambda=0,\end{cases}$$

where $G(x)$ is a cumulative distribution function (cdf) and $\lambda \in \mathbb{R}$ is a constant. The cdf F is called exp-G distribution. They obtained several mathematical properties of this class of distributions and discussed the two special cases: exp-Weibull and exp-beta distributions. Recently, Javanshiri et al. (2013) introduced exp-uniform (EU) distribution by taking $G(x)$ to be the cdf of the uniform distribution with parameters a and b and studied its properties and application.

In this paper, we extend the idea of $T-X$ family of distributions to introduce our new class as follows:

Let $F(t)$ be the cdf of a random variable T and $G(x)$ be the cdf of a random variable X defined on $\mathbb{R}$. We define the cdf of $T-X$ family of distributions by

$$H(x)=\frac{F(G(x))-F(0)}{F(1)-F(0)}, \qquad x\in\mathbb{R}.$$

Note that $H(x)$ is a cdf and for the special case $F(x)=1-e^{-\lambda x}$, the distribution is simplified to the previous one. When X is a continuous random variable, the probability density function (pdf) and hazard function of this family are given, respectively, by

$$h(x)=\frac{g(x)f(G(x))}{F(1)-F(0)},$$
$$r(x)=\frac{g(x)f(G(x))}{F(1)-F(G(x))},$$

where $g(x)$ is pdf corresponding to $G(x)$. The rest of the paper is organized as follows. In section 2, the Gamma-Weibull distribution is introduced. Properties of this distribution are obtained in Section 3. In Section 4, the maximum likelihood estimations are discussed. Characterizations are presented in section 5. The proposed model is applied to a data set in Section 6. Concluding remarks are given in section 7.

2. The Generalized Gamma-Weibull Distribution

The Weibull distribution is a popular distribution for modelling lifetime data as well as modelling phenomenon with monotone failure rates. The two-parameter Weibull density function is usually expressed as follows:

$$g(x;\mu,\sigma)=\frac{\mu}{\sigma}x^{\mu-1}\exp\left[-\left(\frac{x}{\sigma}\right)^{\mu}\right], \quad x>0,$$

where $\sigma>0$ is a scale parameter and $\mu>0$ is a shape parameter and its cumulative distribution function is

$$G(x;\mu,\sigma)=1-\exp\left[-\left(\frac{x}{\sigma}\right)^{\mu}\right], \quad x>0.$$

Considering the following Gamma density function

$$f(x;\alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right), \quad x>0,$$

the cdf of the GGW distribution is

$$H(x;\alpha,\beta,\mu,\sigma)=\frac{\Gamma(\alpha)-\Gamma\left(\alpha,\dfrac{1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]}{\beta}\right)}{\Gamma(\alpha)-\Gamma\left(\alpha,\dfrac{1}{\beta}\right)},\qquad x>0,$$

and the corresponding pdf is

$$h(x;\alpha,\beta,\mu,\sigma)=\frac{\mu\left(\dfrac{x}{\sigma}\right)^{\mu}\left(\dfrac{1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]}{\beta}\right)^{\alpha}\exp\left\{\dfrac{\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1}{\beta}\right\}}{x\left(\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1\right)\left[\Gamma(\alpha)-\Gamma\left(\alpha,\dfrac{1}{\beta}\right)\right]},\qquad x>0,$$

where $\Gamma(a,z)=\int_z^{\infty}t^{a-1}e^{-t}dt$ denotes the incomplete gamma function, $\beta,\sigma>0$ are scale parameters and $\alpha,\mu>0$ are shape parameters. A random variable X which follows the GGW distribution with parameters α,β,μ and σ is denoted by $X\sim GGW(\alpha,\beta,\mu,\sigma)$.

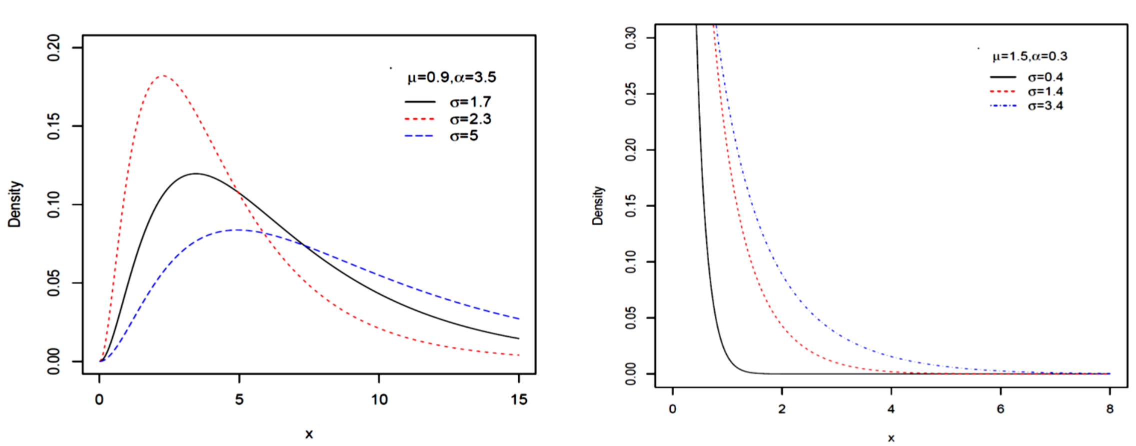

Figure 1: Plots of the pdf of the $GGW(\alpha,1,\mu,\sigma)$ distribution for various values of parameters.

The associated survival and hazard functions of GGW distribution are, respectively, given by

$$S(x)=\frac{\Gamma\left(\alpha,\dfrac{1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]}{\beta}\right)-\Gamma\left(\alpha,\dfrac{1}{\beta}\right)}{\Gamma(\alpha)-\Gamma\left(\alpha,\dfrac{1}{\beta}\right)},$$

and

$$r(x)=-\frac{\mu\beta^{-\alpha}\sigma^{-\mu}x^{\mu-1}\left(1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]\right)^{\alpha}\exp\left[\dfrac{\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1}{\beta}\right]}{\left(\exp\left[\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1\right)\left[\Gamma\left(\alpha,\dfrac{1}{\beta}\right)-\Gamma\left(\alpha,\dfrac{1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]}{\beta}\right)\right]}.$$

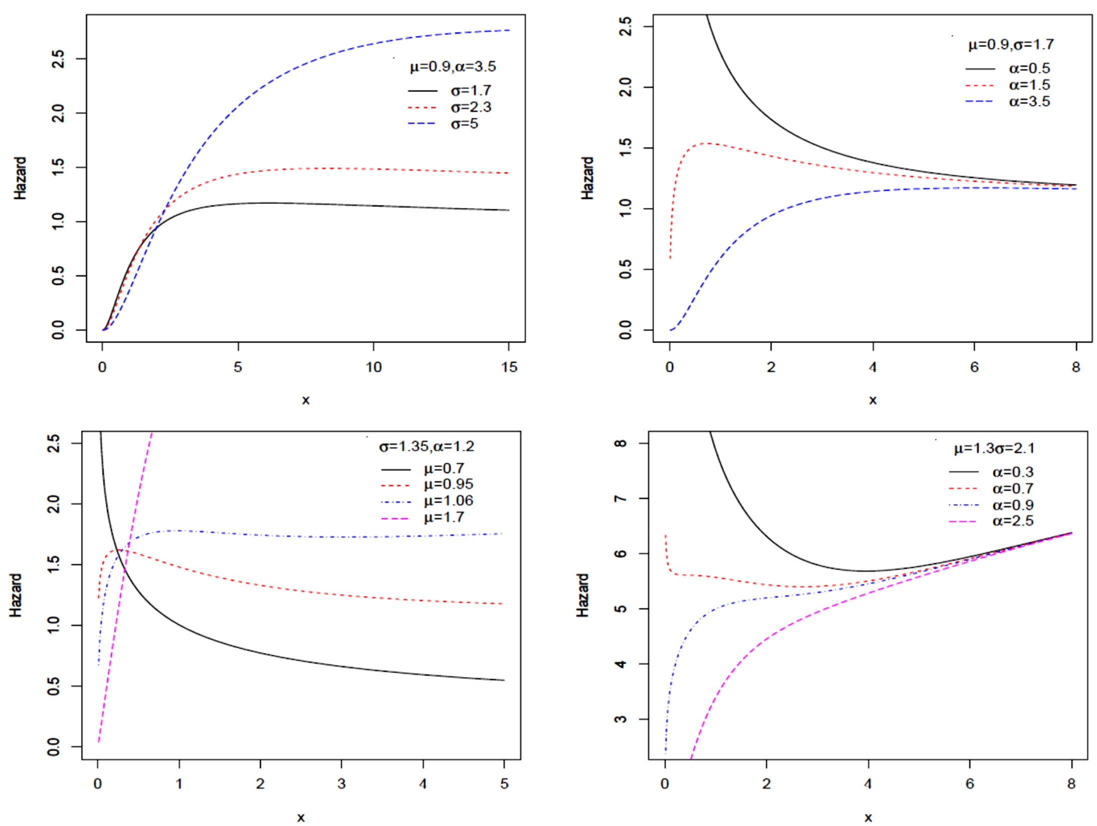

Figure 2: Plots of the hazard rate function of the $GGW(\alpha,1,\mu,\sigma)$ distribution for various values of paramet

Due to the complicated form of $r(x)$, it is not possible to derive its properties mathematically. We make some observations via certain plots of $r(x)$. For some values of the parameters, the plots of the density and the hazard rate function are shown in Figures 1 and 2, respectively.

Some of the shape properties of the GGW distribution can be summarized as follows:

- The distribution is right-skewed when $\alpha > 0.5$. As σ increases the degree of right skewness increases.
- The distribution is reverse "J" shaped when $\mu < 1, \alpha < 1$, also when $\mu > 1, \alpha < 0.5$.

Figure 2 shows that the hazard rate function of the GGW distribution can take monotonic, bathtub and unimodal-bathtub shapes for different parametric combinations.

3. Moments

In general, k -th non-central moment of a GGW distribution cannot be easily evaluated. Considering common definition, k -th non-central moment can be written as

$$E[X^k] = \frac{\mu x^{\mu+k-1}\left(\dfrac{1-\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]}{\beta}\right)^{\alpha}\exp\left\{\dfrac{\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1}{\beta}\right\}}{\sigma^{\mu}\left(\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\mu}\right]-1\right)\left[\Gamma(\alpha)-\Gamma\left(\alpha,\dfrac{1}{\beta}\right)\right]}.$$

These measures for the $GGW(\alpha, \beta, 0.5, 0.7)$ distribution are calculated and presented in Table 1 for various values of α and β.

Table 1: Mean, second moments, variance, skewness and kurtosis when $\mu = 0.5$ and $\sigma = 0.7$ for various values of α and β.

α	β	Mean	$E[X^2]$	Var(X)	Skewness	Kurtosis
0.5	0.1	0.007	0.002	0.002	197.803	144131.7
	0.5	0.245	1.459	1.399	16.608	545.097
	1	0.439	3.053	2.861	11.933	276.483
	2	0.581	4.315	3.977	10.228	201.946
1	0.1	0.022	0.012	0.011	106.469	36134.95
	0.5	0.592	3.947	3.596	10.556	218.449
	1	0.934	7.073	6.201	8.205	130.650
	2	1.152	9.211	7.884	7.333	104.097
3	0.1	0.189	0.407	0.371	26.021	1595.047
	0.5	2.238	19.597	14.589	5.458	58.184
	1	2.762	26.047	18.419	4.902	46.962
	2	3.033	29.558	20.361	4.680	42.851

From Table 1, it can be concluded that for fixed α, the mean, the second moment and the variance are increasing functions of β, while the skewness and the kurtosis are decreasing functions of β. Also, for fixed β, the mean, the second moment and the variance are increasing functions of α, while the skewness and the kurtosis are decreasing functions of α. Table 1 also shows that the GGW distribution is right skewed. Over-dispersion in a distribution is a situation in which the variance exceeds the mean, under-dispersion is the opposite, and equi-dispersion occurs when the variance is equal to the mean. From Table 1, the GGW distribution satisfies the over-dispersion property for almost all values of the parameters.

4. Parameter estimation and inference

Let $X_1,\cdots,X_n$ be a random sample, with observed values $x_1,\cdots,x_n$ from $GGW\,(\alpha,\beta,\mu,\sigma)$. The log-likelihood function for the vector of parameters $\Theta=(\alpha,\beta,\mu,\sigma)^T$ can be written as

$$\begin{aligned}
l_n = l_n(\Theta) = {} & \alpha\sum_{i=1}^{n}\log\left(\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]-1\right)+\sum_{i=1}^{n}\left(\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]-1\right)\\
& -\sum_{i=1}^{n}\log\left(\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]-1\right)+\mu\sum_{i=1}^{n}\log\left(\frac{x_i}{\sigma}\right)-\sum_{i=1}^{n}\log\left(x_i\right)\\
& -n\log\left[\Gamma(\alpha)-\Gamma(\alpha,\frac{1}{\beta})\right]+n\log(\mu).
\end{aligned}$$

The components of the score vector $U(\Theta)=\left(\frac{\partial l_n}{\partial\alpha},\frac{\partial l_n}{\partial\beta},\frac{\partial l_n}{\partial\mu},\frac{\partial l_n}{\partial\sigma}\right)^T$ are given by

$$\frac{\partial l_n}{\partial\alpha}=\sum_{i=1}^{n}\log\left(\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]-1\right)-\frac{n\,\psi(\alpha)\Gamma(\alpha)-n\,G_{2,3}^{3,0}\left(\frac{1}{\beta}\,\middle|\,\begin{matrix}1,1\\0,0,\alpha\end{matrix}\right)-n\log\left(\frac{1}{\beta}\right)\Gamma\left(\alpha,\frac{1}{\beta}\right)}{\Gamma(\alpha)-\Gamma\left(\alpha,\frac{1}{\beta}\right)},$$

$$\frac{\partial l_n}{\partial\beta}=\frac{e^{-1/\beta}n\left(\frac{1}{\beta}\right)^{\alpha+1}}{\Gamma(\alpha)-\Gamma\left(\alpha,\frac{1}{\beta}\right)},$$

$$\begin{aligned}
\frac{\partial l_n}{\partial\mu}= {} & \frac{n}{\mu}-(\alpha-1)\sum_{i=1}^{n}\frac{\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]\left(\frac{x_i}{\sigma}\right)^{\mu}\log\left(\frac{x_i}{\sigma}\right)}{e^{-\left(\frac{x_i}{\sigma}\right)^{\mu}}-1}\\
& -\sum_{i=1}^{n}\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]\left(\frac{x_i}{\sigma}\right)^{\mu}\log\left(\frac{x_i}{\sigma}\right)+\sum_{i=1}^{n}\log\left(\frac{x_i}{\sigma}\right),
\end{aligned}$$

$$\frac{\partial l_n}{\partial \sigma} = (\alpha-1)\sum_{i=1}^{n} \frac{\mu x_i \exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]\left(\frac{x_i}{\sigma}\right)^{\mu-1}}{\sigma^2\left(\exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]-1\right)} + \sum_{i=1}^{n} \frac{\mu x_i \exp\left[-\left(\frac{x_i}{\sigma}\right)^{\mu}\right]\left(\frac{x_i}{\sigma}\right)^{\mu-1}}{\sigma^2} - \frac{\mu n}{\sigma},$$

where $\psi(.)$ is the digamma function and $G_{pq}^{mn}(.)$ is the Meijer G-function. The maximum likelihood estimation (MLE) of Θ, say $\hat{\Theta}$, is obtained by solving the nonlinear system $U(\Theta)=\mathbf{0}$. This nonlinear system of equations does not have a closed form solution.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta}-\Theta)$ is $N_4(\mathbf{0}, I^{-1}(\Theta))$ where $I^{-1}(\Theta)$ is the inverse of the Fisher information matrix. A $100(1-\gamma)$ % asymptotic confidence interval (ACI) for each parameter Θ_i is given by

$$AIC(\Theta_i) = \left(\hat{\Theta}_i - z_{\alpha/2}\sqrt{\hat{I}^{-1}(\hat{\Theta})_i / n}, \hat{\Theta}_i + z_{\alpha/2}\sqrt{\hat{I}^{-1}(\hat{\Theta})_i / n}\right),$$

where $\hat{I}^{-1}(\hat{\Theta})_i$ is the i -th diagonal element of $\hat{\mathbf{I}}^{-1}(\hat{\Theta})$.

5. Characterizations

The problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. In designing a stochastic model for a particular modelling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. The investigator, therefore, will rely on the characterizations of the selected distribution. These characterizations have been established in many different directions. The present work deals with the characterizations of GGW distribution which are based on a simple relationship between two truncated moments. Our characterization results presented here will employ an interesting result due to Glanzel (1987) (Theorem 5.1 below). The advantage of the characterizations given here is that, cdf H need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation. We believe that other characterizations of GGW distribution may not be possible due to the structure of its cdf.

Theorem 5.1. *Let* $(\Omega, \mathrm{F}, \mathbf{P})$ *be a given probability space and let* $I=[a,b]$ *be an interval for some* $a<b$ $(a=-\infty, b=\infty$ *might as well be allowed). Let* $X:\Omega \to I$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* I *such that*

$$\mathbf{E}\left[q_1(X) \mid X \geq x\right] = \mathbf{E}\left[q_2(X) \mid X \geq x\right]\eta(x), \qquad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(I), \eta \in C^2(I)$ and F is twice continuously differentiable and strictly monotone function on the set I. Finally, assume that the equation $q_2\eta = q_1$ has no real solution in the interior of I. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_2(u) - q_1(u)}\right| \exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta' q_2}{\eta q_2 - q_1}$ and C is a constant, chosen to make $\int_I dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1,n}, q_{2,n}$ and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 5.1 and let $q_{1,n} \to q_1, q_{2,n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1,n}(X)$ and $q_{2,n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E[q_1(X) \mid X \geq x]}{E[q_2(X) \mid X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and η, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Levy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glanzel and Hamedani (2001),

Remark 5.2.

(a) *In Theorem 5.1, the interval I need not be closed since the condition is only on the interior of I.*

(b) *Clearly, Theorem 5.1can be stated in terms of two functions q_1 and η by taking $q_2(x) \equiv 1$, provided that the cdf F has a closed form, which will reduce the condition given in Theorem 5.1 to $E[q_1(X) \mid X \geq x] = \eta(x)$. However, adding an extra function will give a lot more flexibility, as far as its application is concerned.*

Proposition 5.3.

Let $X : \Omega \to (0,\infty)$ *be a continuous random variable and let* $q_2(x) = \left(1 - e^{-(x/\sigma)^{\mu}}\right)^{1-\alpha}$ *and* $q_1(x) = q_2(x)\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right)$ *for* $x \in (0,\infty)$. *Then* X *has pdf* $h(x)$ *if and only if the function* η *defined in Theorem 5.1 has the form*

$$\eta(x) = \frac{1}{2}\left\{1 + \exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right)\right\}, \qquad x > 0.$$

Proof. Let X have density $h(x)$, then

$$\left(1 - H(x)\right)\mathbf{E}\left[q_2(X) | \geq x\right] = \frac{\beta^{-(\alpha-1)} e^{-1/\beta}}{K}\left[\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right) - 1\right], \qquad x > 0,$$

and

$$\left(1 - H(x)\right)\mathbf{E}\left[q_1(X) | X \geq x\right] = \frac{\beta^{-(\alpha-1)} e^{-1/\beta}}{2K}\left[\exp\left(\frac{2}{\beta} e^{-(x/\sigma)^{\mu}}\right) - 1\right], \qquad x > 0,$$

where $K = \Gamma(\alpha) - \Gamma\left(\alpha, \frac{1}{\beta}\right)$.

Finally

$$\eta(x) q_2(x) - q_1(x) = \frac{1}{2} q_2(x)\left\{1 - \exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right)\right\} < 0 \quad \textit{for} \quad x > 0.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_2(x)}{\eta(x) q_2(x) - q_1(x)} = \frac{\mu x^{\mu-1} e^{-(x/\sigma)^{\mu}} \exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right)}{\beta\sigma^{\mu}\left[\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right) - 1\right]}, \quad x > 0,$$

and hence

$$s(x) = -\ln\left\{\frac{\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^{\mu}}\right) - 1}{e^{\frac{1}{\beta}} - 1}\right\}, \qquad x > 0.$$

Now, in view of Theorem 5.1, X has density $h(x)$.

Corollary 5.4. *Let $X:\Omega \to (0,\infty)$ be a continuous random variable and let $q_2(x)$ be as in Proposition 5.3. The pdf of X is $h(x)$ if and only if there exist functions q_1 and η defined in Theorem 5.1satisfying the differential equation*

$$s'(x) = \frac{\mu x^{\mu-1} e^{-(x/\sigma)^\mu} \exp\left(\frac{1}{\beta} e^{-(x/\sigma)^\mu}\right)}{\beta \sigma^\mu \left[\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^\mu}\right) - 1\right]}, \qquad x > 0.$$

Remark 5.5.

(a) *The general solution of the differential equation in Corollary 5.4 is*

$$\eta(x) = \left[\exp\left(\frac{1}{\beta} e^{-(x/\sigma)^\mu}\right) - 1\right]^{-1} \times \left[-\int \mu \beta \sigma^{-\mu} x^{\mu-1} e^{-(x/\sigma)^\mu} \exp\left(\frac{1}{\beta} e^{-(x/\sigma)^\mu}\right) \left(q_2(x)\right)^{-1} q_1(x) dx + D\right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 5.3 with $D = -\frac{1}{2}$.

(b) *Clearly there are other triplets of functions (q_1, q_2, η) satisfying the conditions of Theorem 5.1. We presented one such triplet in Proposition 5.3.*

6. Applications

In this section, the flexibility and applicability of the proposed model is illustrated as compared to the alternative Gamma-Weibull (GW.a) distribution introduced by Alzaatre at al. (2014) and Gamma-Weibull (GW.p) distribution presented by Provost et al. (2011). The Gamma-Weibull model is applied to a data set published in Suprawhardana et al. (1999) which consists of time between failures (thousands of hours) of secondary reactor pumps. The data set consists of 23 observations. The TTT plot of this set of data in Figure 3 displays a bathtub-shaped hazard rate function that indicates the appropriateness of the GGW distribution to fit the data set.

In order to compare the models, the MLEs of the parameters, -2log-likelihood, the Kolmogorov-Smirnov test statistic (K-S), p-value, the Anderson-Darling test statistic (AD), the Cramer-von Misestest statistic (CM) and Durbin-Watsontest statistic (DW) are given in Table 2 for this data set. The CM and DW test statistics are described in details in Chen and Balakrishnan (1995) and Watson (1961), respectively. In general, the smaller the values of K-S, AD, CM and WA, the better the fit to the data. From the values of these statistics, we conclude that the GGW distribution provides a better fit to this data set than the other models.

The fitted densities and the empirical distributions versus the fitted cumulative distribution functions of NP, N and SN models are displayed in Figure 4. These plots suggest that the GGW distribution is superior to the other distributions in terms of the model fitting.

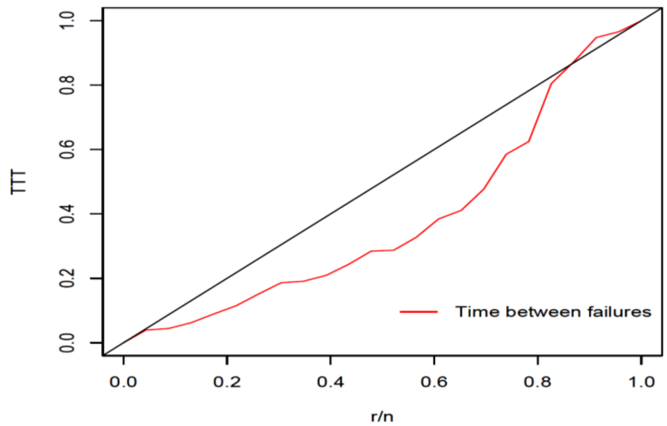

Figure 3: TTT plot to time between failures data

Table 2: MLEs, $-2\log L$, K -S, P-value, AD, CM and DW statistics for time between failures

Dist	MLE	$-2\log L$	K-S	P-value	AD	CM	DW
GGW	$\hat{\alpha}=8.264, \hat{\mu}=0.333,$ $\hat{\sigma}=0.051$	63.639	0.098	0.97	0.233	0.107	0.107
GW.a	$\hat{k}=0.814, \hat{\xi}=0.010,$ $\hat{\lambda}=1.419$	65.045	0.119	0.87	0.406	0.142	0.138
GW.p	$\hat{\alpha}=16.197, \hat{\gamma}=0.054,$ $\hat{c}=0.184, \hat{\beta}=0.102$	63.635	0.099	0.96	0.246	0.110	0.109

7. Conclusions

In this paper, a new method to generate family of distributions is proposed. Then, a family of univariate distributions generated by the Gamma random variable was defined. As an special case, the generalized gamma-Weibull (GGW) distribution is studied and its parameter estimations are considered. Finally, in order to show the usefulness of the new distribution, an application to a real data set is demonstrated. The current study confirms that the proposed distribution, with better flexibility, can be considered to be a great model for real data in comparison with the other competing distributions.

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