The New Kumaraswamy Kumaraswamy Weibull Distribution with Application

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Abstract

The Weibull distribution has been used over the past decades for modeling data in many fields so finding generalization of the Weibull distribution becomes very useful to fit more cases or get better fits than before. In this paper, the Kumaraswamy Kumaraswamy Weibull (Kw Kw W) distribution is presented for the first time and we show that it generalizes many important distributions. The probability density function (pdf), the cumulative distribution function (cdf), moments, quantiles, the median, the mode, the mean deviation, the entropy, order statistics, L-moments, extreme value and parameters estimation based on maximum likelihood are obtained for the Kw Kw W distribution. An illustration study and a real data set are used to illustrate the potentiality and application of the new Kw Kw W distribution.

Keywords: The Weibull distribution, The Kumaraswamy Kumaraswamy family, Moments, Order statistics, L-moments, Entropy, Extreme values, Maximum likelihood estimation.

1. Introduction

Eugene *et al.* (2002) presented for the first time the family of beta generalized distributions which has the following cdf and pdf

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x;W)} t^a (1-t)^b \, dt \;\; ; a,b > 0 \; ; 0 < t < 1 ; -\infty < x < \infty$$

and

$$f(x) = \frac{1}{B(a,b)} g(x;W)\, G(x;W)^{a-1} \left\{ 1 - G(x;W) \right\}^{b-1} \;\; ; a,b > 0 \; ; -\infty < x < \infty$$

Where $G(x;W)$ and $g(x;W)$ are the cdf and the pdf of the baseline distribution, W is parameters vector of the baseline distributions.

Wahed (2006) presented a general method of constructing extended families of distribution from an existing continuous class using the following equation

$$F(x;T,W)=\int_{0}^{G_1(x,W)} g_2(\text{t;T})\,dt\;;0<t<1$$

Where $G_1(x;W)$ and $g_1(x;W)$ are the cdf and pdf of the baseline distribution, $G_2(t;T)$ and $g_2(t;T)$ are the cdf and the pdf of the generator distribution, T and W are respectively parameters vectors of generator and baseline distributions.

Cordeiroa and Castro (2010) presented the family of Kumaraswamy generalized distributions which has the following cdf and pdf

$$F(x;\text{a},\text{b},W)=1-\left\{1-G(x;W)^a\right\}^b\;\;;a,b>0\;;-\infty<x<\infty$$

and

$$f(x;\text{a},\text{b},W)=ab\,g(x)\,G(x)^{a-1}\left\{1-G(x)^a\right\}^{b-1}\;;a,b>0\;;-\infty<x<\infty$$

Where *G(x;W)* and *g(x;W)* are the cdf and the pdf of the baseline distribution, W is parameters vector of the baseline distributions.

Mahmoud *et al.* (2015) presented for the first time the Kumaraswamy Kumaraswamy family of generalized distributions which has the following cdf and pdf

$$F(x;a,b,\alpha,\beta,\underline{\text{W}})=1-\left\{1-\left[1-\left(1-G^{\alpha}\left(x,\underline{\text{W}}\right)\right)^{\beta}\right]^{a}\right\}^{b};-\infty<x<\infty;\;a,b,\alpha,\beta>0 \qquad (1)$$

and

$$f(x;a,b,\alpha,\beta,\underline{\text{W}})=ab\,\alpha\,\beta\,g\,(x,\underline{\text{W}})\,G^{\,\alpha-1}(x,\underline{\text{W}})\,(1-G^{\,\alpha}(x,\underline{\text{W}}))^{\beta-1}\left[1-(1-G^{\,\alpha}(x,\underline{\text{W}}))^{\beta}\right]^{a-1}$$

$$\times\left\{1-\left[1-(1-G^{\,\alpha}(x,\underline{\text{W}}))^{\beta}\right]^{a}\right\}^{b-1};-\infty<x<\infty;\;a,b,\alpha,\beta>0 \qquad (2)$$

Where $G(x;\underline{\text{W}})$ and $g(x;\underline{\text{W}})$ are the cdf and the pdf of the baseline distribution, W is parameters vector of the baseline distributions.

The main idea of this paper is based on deriving the Kw Kw W distribution from the Kumaraswamy Kumaraswamy family of generalized distributions see Mahmoud *et al.*(2015), where the generator distribution is the Kumaraswamy Kumaraswamy distribution, see El-Sherpieny and Ahmed (2014), which has the following cdf

$$G(x)=1-\left(1-x^{\alpha}\right)^{\beta}\;\;;\;\;0<x<1;\;\alpha,\beta>0$$

2. The New Kumaraswamy Kumaraswamy Weibull Distribution

Since the cdf and pdf of the Weibull distribution respectively are

$$G(x;\theta,\lambda)=1-e^{-(\lambda x)^{\theta}};x,\theta,\lambda>0 \tag{3}$$

and

$$g(x;\theta,\lambda)=\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}};x,\theta,\lambda>0 \tag{4}$$

Then, substituting (3) and (4) into (1) and (2) yields the cdf and the pdf of Kw Kw W distribution as follows

$$F(x;M)=1-\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b};x>0;a,b,\alpha,\beta,\theta,\lambda>0 \tag{5}$$

and

$$f(x;M)=ab\,\alpha\,\beta\,\theta\,\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta-1} \tag{6}$$

$$\times\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a-1}\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1}$$

Where $M=(a,b,\alpha,\beta,\theta,\lambda)$, the Kw Kw W distribution has five shape parameters a, b, α, β, θ and has one scale parameter λ.

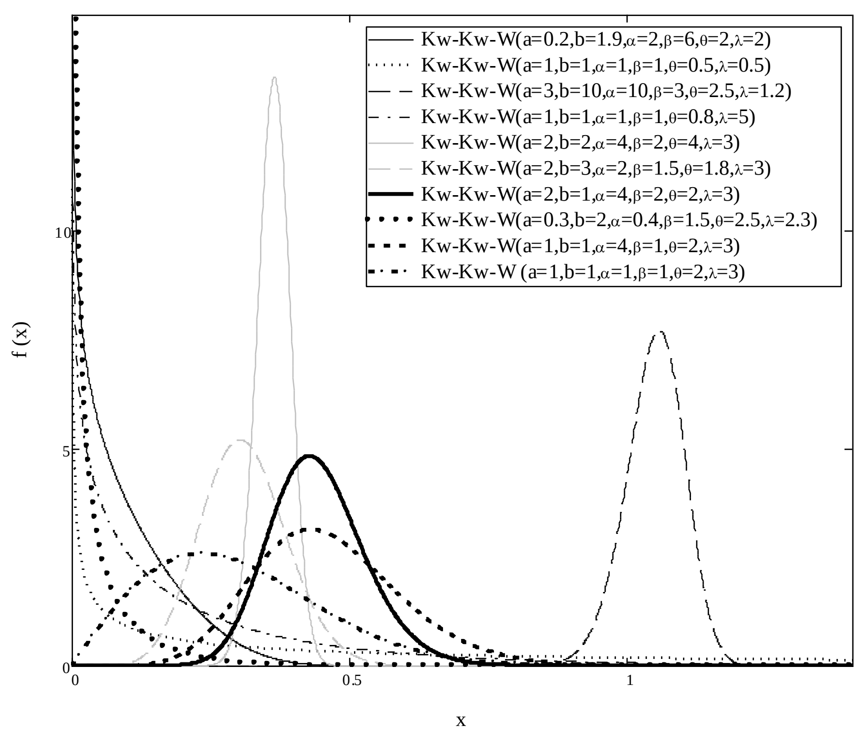

Figure 1: Plot of the Kw Kw W pdf for some parameter values

We see from the last Figure that the sex parameters give a high degree of flexibility for the Kw Kw W distribution.

Some important sub-models of the Kw Kw W distribution are illustrated in the following table

#	Distribution	a	b	α	β	θ	λ
1	Kw Kw exponential	-	-	-	-	1	-
2	Kw Kw Rayleigh	-	-	-	-	2	-
3	Exponentiated Kw W	-	1	-	-	-	-
4	Exponentiated Kw exponential	-	1	-	-	1	-
5	Exponentiated Kw Rayleigh	-	1	-	-	2	-
6	Exponentiated Generalized W	-	1	1	-	-	-
7	Exponentiated Generalized exponential	-	1	1	-	1	-
8	Exponentiated Generalized Rayleigh	-	1	1	-	2	-
9	Kw W	1	1	-	-	-	-
10	Kw exponential	1	1	-	-	1	-
11	Kw Rayleigh	1	1	-	-	2	-
12	Exponentiated W	1	1	-	1	-	-
13	Exponentiated exponential	1	1	-	1	1	-
14	Exponentiated Rayleigh	1	1	-	1	2	-
15	W	1	1	1	1	-	-
16	Exponential	1	1	1	1	1	-
17	Rayleigh	1	1	1	1	2	-

2.1 An Expansion for The pdf

We shall obtain expansion for the pdf of the Kw Kw W distribution as follows:

Using the binomial expansion of the pdf of the Kw Kw W distribution, where b is a real non integer, yields

$$f(x;M) = \theta\lambda^{\theta} \sum_{i,j,k,\ell=0}^{\infty} (1+\ell)\, w_{i,j,k,\ell}\, x^{\theta-1} e^{-(\lambda x)^{\theta}(1+\ell)}$$

Where,

$$w_{i,j,k,\ell} = ab\,\alpha\,\beta(-1)^{i+j+k+\ell} \binom{b-1}{i} \binom{a+ai-1}{j} \binom{\beta j+\beta-1}{k} \binom{\alpha k+\alpha-1}{\ell}$$

Condition for the expansion of the pdf of the Kw Kw W distribution
Since,

$$\int_0^{\infty} f(x;M)\,dx = 1$$

Then,

$$f(x;M) = \theta\lambda^{\theta} \sum_{i,j,k,\ell=0}^{\infty} (1+\ell)\, w_{i,j,k,\ell}\, x^{\theta-1} e^{-(\lambda x)^{\theta}(1+\ell)} \qquad (7)$$

Where,

$$w_{i,j,k,\ell} = ab\,\alpha\,\beta(-1)^{i+j+k+\ell} \binom{b-1}{i} \binom{a+ai-1}{j} \binom{\beta j+\beta-1}{k} \binom{\alpha k+\alpha-1}{\ell}$$

and

$$\sum_{i,j,k,\ell=0}^{\infty} w_{i,j,k,\ell} = 1$$

2.2 An Expansion for The cdf

We shall obtain an expansion for the cdf of the Kw Kw W distribution as follows:

Using binomial expansion in (5) yields

$$F(x;M) = 1 - \sum_{m,n,p,q=0}^{\infty} \binom{b}{m}\binom{a\,m}{n}\binom{\beta n}{p}\binom{\alpha p}{q}(-1)^{m+n+p+q} e^{-q(\lambda x)^{\theta}}$$

Hence,

$$F(x;M) = 1 - \sum_{m,n,p,q=0}^{\infty} M_{m,n,p,q}\, e^{-q(\lambda x)^{\theta}} \tag{8}$$

Where,

$$M_{m,n,p,q} = \binom{b}{m}\binom{a\,m}{n}\binom{\beta n}{p}\binom{\alpha p}{q}(-1)^{m+n+p+q}$$

3. Some Properties of the Kw Kw W distribution

In this section some properties of the Kw Kw W distribution will be obtained as follows:

3.1 The rth Moment

Generally, the rth moment of a random variable X can be given from

$$E\left(X^{r}\right) = \int_{-\infty}^{\infty} x^{r} f(x;M)\,dx$$

Substituting (7) into the last equation yields

$$E\left(X^{r}\right) = \theta\lambda^{\theta} \sum_{i,j,k,\ell=0}^{\infty} (1+\ell)\, w_{i,j,k,\ell} \int_{0}^{\infty} x^{r+\theta-1} e^{-(\lambda x)^{\theta}(1+\ell)}\,dx$$

Hence,

$$E\left(X^{r}\right) = \sum_{i,j,k,\ell=0}^{\infty} w_{i,j,k,\ell} \left(\frac{\lambda^{-\theta}}{1+\ell}\right)^{\frac{r}{\theta}} \Gamma\left(1+\frac{r}{\theta}\right) \tag{9}$$

3.2 The Moment Generating Function

Generally, the moment generating function of a random variable X can be given from

$$M_{x}(t) = \int_{-\infty}^{\infty} e^{tx} f(x;M)\,dx$$

Then,

$$E\left(e^{tx}\right)=\sum_{r=0}^{\infty}\frac{t^{r}}{r!}\int_{-\infty}^{\infty}x^{r}f(x;M)dx$$

Hence,

$$M_{x}(t)=\sum_{i,j,k,\ell=0}^{\infty}w_{i,j,k,\ell}\sum_{r=0}^{\infty}\left(\frac{t}{\lambda(1+\ell)^{1/\theta}}\right)^{r}\frac{1}{r!}\Gamma\left(1+\frac{r}{\theta}\right)$$

Using generalized hypergeometric function definition

$${}_{p}\psi_{q}\left[\begin{matrix}(\gamma_1,\mathrm{A}_1),\ldots,(\gamma_p,\mathrm{A}_p)\\(\phi_1,\mathrm{B}_1),\ldots,(\phi_q,\mathrm{B}_q)\end{matrix};\mathrm{y}\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\gamma_j+A_j n)y^{n}}{\prod_{j=1}^{q}\Gamma(\phi_j+B_j n)n!}$$

Then,

$${}_{1}\psi_{0}\left[\begin{matrix}(\gamma_1,\mathrm{A}_1)\\-\end{matrix};\mathrm{y}\right]=\sum_{n=0}^{\infty}\frac{y^{n}\Gamma(\gamma_1+A_1 n)}{n!}$$

Hence,

$$M_{x}(t)=\sum_{i,j,k,\ell=0}^{\infty}w_{i,j,k,\ell}\;{}_{1}\psi_{0}\left[\begin{matrix}(1,1/\theta)\\-\end{matrix};\frac{t}{\lambda(1+\ell)^{1/\theta}}\right]$$

3.3 Quantile Function

Starting with the well-known definition of the 100 q^{th} quantile it is clear that

$$q=P(X\le x_q)=F(x_q;\mathrm{M});\quad x_q>0,\ 0<q<1$$

Quantiles of the Kw Kw W distribution can be obtained by equating the cdf from (5) to q:

$$q=1-\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b}$$

Then,

$$x=\left\{-\lambda^{-\theta}\log\left\{1-\left[1-\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}\right\}^{\frac{1}{\theta}}$$

Easily, substituting q with 1/2 (the second quantile) yields the median

3.4 The Mode

The nature logarithm of (6) is

$$I(x;M)=\ln\left(ab\,\alpha\,\beta\,\theta\,\lambda^{\theta}\right)+(\theta-1)\ln(x)-\left[\lambda(x)\right]^{\theta}+(\alpha-1)\ln\left[1-e^{-\left[\lambda.(x)\right]^{\theta}}\right]$$
$$+(\beta-1)\ln\left[1-\left[1-e^{-\left[\lambda.(x)\right]^{\theta}}\right]^{\alpha}\right]+(a-1)\ln\left[1-\left[1-\left[1-e^{-\left[\lambda(x)\right]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]$$
$$+(b-1)\left[1-\left[1-\left[1-\left[1-e^{-\left[\lambda(x)\right]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}\right]$$

Then, differentiating the nature logarithm of (6) with respect to *x* and equating it to zero yields:

$$\frac{\theta-1}{x}-\theta\lambda(\lambda x)^{\theta-1}-\frac{\theta\lambda e^{-[\lambda x]^{\theta}}(\alpha-1)(\lambda x)^{\theta-1}}{e^{-[\lambda.x]^{\theta}}-1}+\frac{\alpha\theta\lambda e^{-[\lambda x]^{\theta}}(\beta-1)\left[1-e^{-(\lambda x)^{\theta}}\right]^{\alpha-1}(\lambda x)^{\theta-1}}{\left[1-e^{-(\lambda x)^{\theta}}\right]^{\alpha}-1}$$
$$-\frac{\beta\alpha\theta\lambda e^{-(\lambda x)^{\theta}}\left[1-\left[1-e^{-(\lambda x)^{\theta}}\right]^{\alpha}\right]^{\beta-1}(a-1)\left[1-e^{-(\lambda x)^{\theta}}\right]^{\alpha-1}(\lambda x)^{\theta-1}}{\left[1-e^{-(\lambda x)^{\theta}}\right]^{\alpha}-1}$$
$$+\frac{a.\beta.\alpha.\theta.\lambda.e^{-(\lambda.x)^{\theta}}\left[1-\left[1-e^{-(\lambda.x)^{\theta}}\right]^{\alpha}\right]^{\beta-1}.(b-1).\left[1-e^{-(\lambda.x)^{\theta}}\right]^{\alpha-1}}{\left[1-\left[1-\left[1-e^{-(\lambda.x)^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1}$$
$$\times(\lambda.x)^{\theta-1}\left[1-\left[1-\left[1-e^{-(\lambda.x)^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a-1}=0$$

The last equation is a nonlinear equation and does not have an analytic solution with respect to *x*. Therefore, we have to solve it numerically. If x_0 is a root for the last equation it must be $\frac{\partial^2}{\partial x^2}\log \mathrm{f}(x_0)<0$

3.5 The Mean Deviation

Generally, the mean deviation about the mean and about the median for a random variable X respectively can be given from

$$\delta_1(X)=\int_{-\infty}^{\infty}|x-\mu|\,f(x)dx \text{ and } \delta_2(X)=\int_{-\infty}^{\infty}|x-\eta|\,f(x)dx$$

Easily, we can obtain them respectively from

$$\delta_1(X) = 2\mu F(\mu) - 2J(\mu) \text{ and } \delta_2(X) = \mu - 2J(\eta)$$

In order to obtain $\delta_1(X)$ and $\delta_2(X)$ we must obtain the following integration of *J*(.), where *J*(.) is called the incomplete first moment

$$J(Z) = \int_0^z x\, f(x;M)\, dx$$

Substituting (7) into the last equation yields

$$J(x;M) = \theta\lambda^\theta \sum_{i,j,k,\ell=0}^{\infty} (1+\ell)\, w_{i,j,k,\ell} \int_0^z x^\theta e^{-(\lambda x)^\theta (1+\ell)} dx$$

Hence,

$$J(z,M) = \sum_{i,j,k,\ell=0}^{\infty} \frac{w_{i,j,k,\ell}}{\lambda(1+\ell)^{\frac{1}{\theta}}} \Gamma(\frac{1}{\theta}+1, (1+\ell)(\lambda z)^\theta)$$

4. The Survival and Hazard Function

Generally, the survival function of a random variable X can be given from

$$S(x;M) = 1 - F(x;M)$$

Substituting (5) into last equation yields

$$S(x;M) = \left\{1 - \left[1 - \left(1 - \left(1 - e^{-(\lambda x)^\theta}\right)^\alpha\right)^\beta\right]^a\right\}^b \qquad (10)$$

Generally, the Hazard function can be given from

$$H(x;M) = \frac{f(x;M)}{S(x;M)}$$

so the hazard function of the Kw Kw W distribution can be given by substituting (6) and (10) into last equation

$$H(x;M) = \frac{ab\,\alpha\beta\theta\lambda^\theta x^{\theta-1} e^{-(\lambda\theta)^\theta}\left(1-e^{-(\lambda x)^\theta}\right)^{\alpha-1}\left(1-\left(1-e^{-(\lambda x)^\theta}\right)^\alpha\right)^{\beta-1}\left[1-\left(1-\left(1-e^{-(\lambda x)^\theta}\right)^\alpha\right)^\beta\right]^{a-1}}{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^\theta}\right)^\alpha\right)^\beta\right]^a}$$

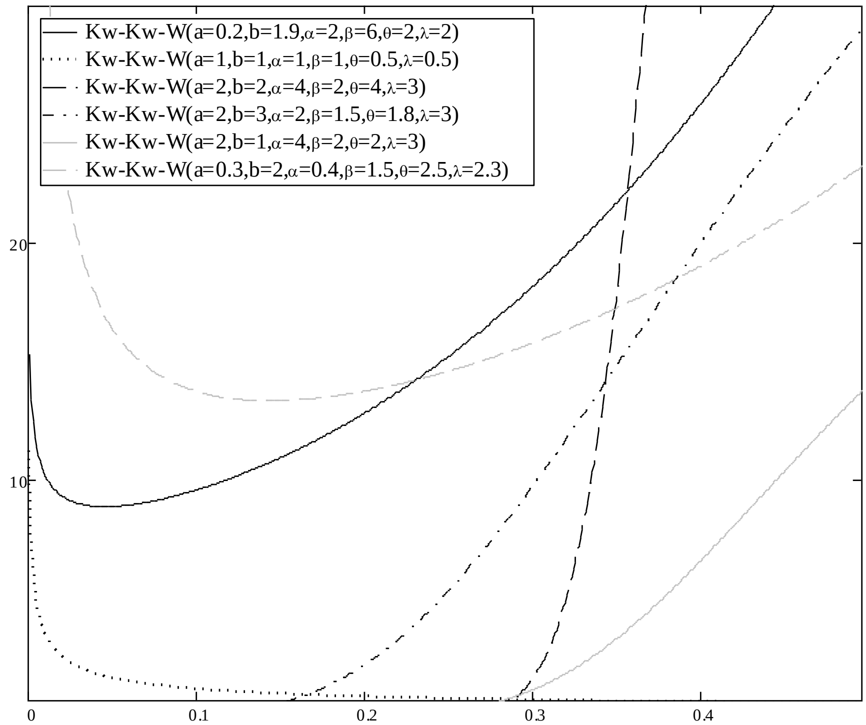

Figure 2: Plot of the Kw Kw W Hazard for some parameters values

We see from the last Figure that the Hazard rate function of the Kw Kw W distribution has many shapes like decreasing shapes, increasing shapes and decreasing then increasing shapes.

5. Order Statistics

A simple random sample $X_1, X_2, .., X_v$ given from the Kw Kw W distribution where X´s are i.i.d random variables, has the density $f_{u:v}(x_{u:v})$ of the u^{th} order statistic, for u =1,2,......,v

as follows, see Arnold, *et al*.(1992)

$$f_{u:v}(x_{u:v}) = \frac{f(x_u)}{B(u, v-u+1)} F(x_u)^{u-1} \left\{ 1 - F(x_u) \right\}^{v-u} \tag{14}$$

Since,

$$\left[1 - F(x_u)\right]^{v-u} \left[F(x_u)\right]^{u-1} = \sum_{m=0}^{u-1} \sum_{n,p,q,s=0}^{\infty} N_{m,n,p,q,s} e^{-s(\lambda x_u)^{\theta}} \tag{15}$$

Where,

$$N_{m,n,p,q,s} = \binom{u-1}{m} \binom{b(m+v-u)}{n} \binom{an}{p} \binom{\beta p}{q} \binom{\alpha q}{s} (-1)^{m+n+p+q+s}$$

Substituting (15) and (7) into (14) yields the density $f_{u:v}(x_{u:v})$ of the u[th] order statistic

$$f_{u:v}(x_{u:v};M)=\frac{1}{B(u,v-u+1)}\theta\lambda^{\theta}\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{u-1}\sum_{n,p,q,s=0}^{\infty}\left(1+\ell\right)w_{i,j,k,\ell} \times N_{m,n,p,q,s}\,x_u^{\theta-1}e^{-(\lambda x_u)^{\theta}(1+\ell+s)} \quad (16)$$

Moments of order statistics

Generally, the moments of order statistics can be given from

Substituting (16) into last equation gives

$$E_{u:v}(X_{u:v}^{r})=\frac{1}{B(u,v-u+1)}\theta\lambda^{\theta}\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{u-1}\sum_{n,p,q,s=0}^{\infty}\left(1+\ell\right)w_{i,j,k,\ell}N_{m,n,p,q,s} \quad (17)$$

$$\times\int_0^{\infty}x_u^{r+\theta-1}e^{-(\lambda x_u)^{\theta}(1+\ell+s)}dx_u$$

Let $I=\int_0^{\infty}x_u^{r+\theta-1}e^{-(\lambda x_u)^{\theta}(1+\ell+s)}dx_u$

Then,

$$I=\frac{\lambda^{-r-\theta}}{\theta\left(1+\ell+s\right)^{1+\frac{r}{\theta}}}\Gamma\left(1+\frac{r}{\theta}\right) \quad (18)$$

Substituting (18) into (17) yields

$$E_{u:v}(X_{u:v}^{r})=\frac{1}{B(u,v-u+1)}\sum_{i,j,k,l=0}^{\infty}\sum_{m=0}^{u-1}\sum_{n,p,q,s=0}^{\infty}\left(1+\ell\right)w_{i,j,k,\ell}N_{m,n,p,q,s}\frac{\lambda^{-r}}{\left(1+\ell+s\right)^{1+\frac{r}{\theta}}}\Gamma\left(1+\frac{r}{\theta}\right)$$

L-moments are linear functions of expected order statistics where L-moments is defined as, see Hosking (1990):

$$\lambda_{\alpha+1}=(\alpha+1)^{-1}\sum_{t=0}^{\alpha}(-1)^{t}\binom{\alpha}{t}E(x_{\alpha+1-t:\alpha+1})\ ,\alpha=0,1,....$$

the first L-moments is

$$\lambda_1=E(X_{1:1})$$

Where,

$$E_{1:1}(X_{1:1}^{r})=\sum_{i,j,k,l=0}^{\infty}\sum_{n,p,q,s=0}^{\infty}\left(1+\ell\right)w_{i,j,k,\ell}N_{n,p,q,s}\frac{\lambda^{-r}}{\left(1+\ell+s\right)^{1+\frac{r}{\theta}}}\Gamma\left(1+\frac{r}{\theta}\right)$$

Similarly λ_2, λ_3 and λ_4 can be obtained.

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - 2X_{1:2})$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$

6. The Asymptotes of *F*(*X*) and *f* (*X*)

In this section, we shall obtain the asymptotes of the cdf and pdf of the Kw Kw W distribution

6.1 The Asymptotes of cdf

First: when x converges to zero

Using Maclaurin series expansion of the cdf gives

$$\lim_{x\to 0} F(x;M) \approx 1-\left\{1-\left(1-\left(1-(\lambda x)^{\theta\alpha}\right)^{\beta}\right)^{a}\right\}^{b}$$

and using binomial expansion gives

$$\lim_{x\to 0} F(x;M) \approx b\beta^{a}(\lambda x)^{\theta\alpha a} \tag{19}$$

Second: when x converges to ∞

Using binomial expansion of the cdf gives

Then,

$$\lim_{x\to\infty} F(x;M) \approx 1-\left[a\alpha^{\beta}e^{-\beta(\lambda x)^{\theta}}\right]^{b}$$

Hence,

$$1-\lim_{x\to\infty} F(x;M) \approx a^{b}\alpha^{b\beta}e^{-b\beta(\lambda x)^{\theta}} \tag{20}$$

6.2 The Asymptotes of the pdf

First: when x converges to zero.

$$\lim_{x\to 0} f(x;M) \approx ab\alpha\beta\theta\lambda^{\theta}x^{\theta-1}\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha-1}\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a-1}$$

Using Maclaurin series expansion of the cdf gives

$$\lim_{x\to 0} f(x;M) = ab\alpha\beta\theta\lambda^{\theta}x^{\theta-1}(\lambda x)^{\theta(\alpha-1)}\left(1-\left(1-(\lambda x)^{\alpha\theta}\right)^{\beta}\right)^{\alpha-1}$$

and using only first and second terms of cdf binomial expansion gives

$$\lim_{x\to 0} f\left(x;M\right) \approx ab\alpha\beta^{a}\theta\lambda^{\alpha a\theta}x^{\alpha a\theta-1}$$

Second: when x converges to ∞

$$\lim_{x\to\infty} f\left(x;M\right) \approx ab\alpha\beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}}\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta-1}\left\{1-\left[1-\left(1-\left(1-e^{-(\lambda x)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1}$$

Using binomial expansion gives

$$\lim_{x\to\infty} f\left(x;M\right) \approx a^{b}b\alpha^{\beta b-1}\beta\theta\lambda^{\theta}x^{\theta-1}e^{-(\lambda x)^{\theta}(\beta+b-1)}$$

7. Extreme Values

If $X_1, X_2, \ldots, X_n$ is a random sample from (6) and if $\bar{X} = \left(X_1 + X_2 + \ldots + X_n\right)/n$ denotes the sample mean, then according to central limit theorem the quantity $\sqrt{n}\left(\bar{X} - E\left(X\right)\right)/\sqrt{Var\left(X\right)}$ approaches the standard normal distribution where $n\to\infty$ In this section, we will obtain some properties of the asymptotes of the extreme values $M_n=max(X_1, X_2, \ldots, X_n)$ and $m_n=min(X_1, X_2, \ldots, X_n)$.

First Minimum value:

It is known from (19) that

$$\lim_{x\to 0}\frac{F\left(tx\right)}{F\left(t\right)} = \frac{\left(\lambda t\right)^{\theta\alpha a}x^{\theta\alpha a}}{\left(\lambda t\right)^{\theta\alpha a}} = x^{\theta\alpha a}$$

It follows from theorem1.6.2 in Leadbetter *et al.*(1987) that there must be norming constants γ_n and δ_n, such that

$$pr\left\{\gamma_n\left(m_n - \delta_n\right)x\right\} \to e^{-x^{\theta\alpha a}} \text{ As } n\to\infty$$

The form of the norming constants γ_n and δ_n can be determined. For instance, using corollary 1.6.3 in Leadbetter *et al.* (1987), gives

$$\gamma_n = F^{-1}\left(0\right) = 0 \text{ and } \delta_n = F^{-1}\left(\frac{1}{n}\right) - F^{-1}\left(0\right) = \frac{1}{\lambda}\left[\frac{1}{nb\beta^{a}}\right]^{\frac{1}{\theta\alpha a}}$$

Second Maximum value:

Since,

$$\lim_{t\to\infty}\frac{1-F\left(t+xg\left(t\right)\right)}{1-F\left(t\right)} = \lim_{t\to\infty}\frac{e^{-b\beta\lambda^{\theta}t^{\theta}\left(1+\frac{xg(t)}{t}\right)^{\theta}}}{e^{-b\beta(\lambda t)^{\theta}}}$$

and using the binomial expansion yields

$$\lim_{t\to\infty}\frac{1-F\left(t+xg\left(t\right)\right)}{1-F\left(t\right)}=\lim_{t\to\infty}\frac{e^{-b\beta\lambda^{\theta}t^{\theta}\left(1+\theta\frac{xg(t)}{t}\right)}}{e^{-b\beta(\lambda t)^{\theta}}}=\lim_{t\to\infty}e^{\frac{-b\beta\lambda^{\theta}t^{\theta}\theta xg(t)}{t}}$$

Then,

$$\lim_{t\to\infty}\frac{1-F\left(t+xg\left(t\right)\right)}{1-F\left(t\right)}=e^{-x}\ \text{ Where, } g\left(t\right)=\left[\frac{b\beta\lambda^{\theta}t^{\theta}\theta}{t}\right]^{-1}$$

it follows from theorem1.6.2 in Leadbetter *et al.*(1987) that there must be norming constants ψ_n and ω_n, such that

$$pr\left\{\psi_n\left(M_n-\omega_n\right)\le x\right\}\to e^{-e^{-x}}\ \text{ As } n\to\infty$$

The form of the norming constants ψ_n and ω_n can be determined. Using Corollary 1.6.3 in Leadbetter *et al.* (1987) yields that

$$\psi_n=\left[g\left(\omega_n\right)\right]^{-1}=\frac{\theta b\beta\lambda^{\theta}d_n^{\theta}}{d_n}=\theta b\beta\lambda^{\theta}d_n^{\theta-1}\quad\text{and}$$

$$\omega_n=F^{-1}\left(1-\frac{1}{n}\right)=\frac{1}{\lambda}\left[\frac{-1}{b\beta}\ln\frac{\left(1-\left(1-\frac{1}{n}\right)\right)}{a^{b}\alpha^{\beta b}}\right]^{\frac{1}{\theta}}$$

Hence,

$$\omega_n=\frac{1}{\lambda\left(b\beta\right)^{\frac{1}{\theta}}}\left[\ln \mathrm{n}+\ln a^{b}\alpha^{\beta b}\right]^{\frac{1}{\theta}}$$

8. The Maximum Likelihood Estimation

We now determine the maximum likelihood estimates (MLEs) of the parameters of the Kw Kw W distribution from complete samples. Let $x_1,x_2,\ldots,x_n$ be a random sample of size n from the Kw Kw W (x;*M*) distribution, where $M=\left(a,b,\alpha,\beta,\theta,\lambda\right)$

The log-likelihood function for the vector of parameters *(a, b, α, β, θ* and *λ*) can be written as

$$\ell\left(M;x\right)=n\log\left[ab\alpha\beta\theta\lambda^{\theta}\right]+\left(\theta-1\right)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left(\lambda x_i\right)^{\theta}+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(1-e^{-(\lambda x_i)^{\theta}}\right)$$
$$+\left(\beta-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\lambda x_i)^{\theta}}\right)^{\alpha}\right]+\left(a-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-\left(1-e^{-(\lambda x_i)^{\theta}}\right)^{\alpha}\right)^{\beta}\right]$$
$$+\left(b-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-\left(1-\left(1-e^{-(\lambda x_i)^{\theta}}\right)^{\alpha}\right)^{\beta}\right)^{a}\right]$$

The score functions for the parameters *a, b, α, β, θ* and *λ* are given by

$$\frac{\partial}{\partial a}\ell(M;x)=\frac{n}{a}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right] \quad (21)$$

$$+\sum_{i=1}^{n}\frac{(b-1)\log\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}}{\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1},$$

$$\frac{\partial}{\partial b}\ell(M;x)=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}\right], \quad (22)$$

$$\frac{\partial}{\partial \beta}\ell(M;x)=\sum_{i=1}^{n}\log\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]+\frac{n}{\beta}+(a-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\log\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]}{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}-1} \quad (23)$$

$$-a(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a-1}\log\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]}{\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1},$$

$$\frac{\partial}{\partial \alpha}\ell(M;x)=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-[\lambda x_i]^{\theta}}\right]+(\beta-1)\sum_{i=1}^{n}\frac{\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\log\left[1-e^{-[\lambda x_i]^{\theta}}\right]}{\left[1-e^{-[\lambda x]^{\theta}}\right]^{\alpha}-1} \quad (24)$$

$$-\beta(a-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\log\left[1-e^{-[\lambda x_i]^{\theta}}\right]}{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}-1}$$

$$+a\beta(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\log\left[1-e^{-[\lambda x_i]^{\theta}}\right]}{\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1}$$

$$\times\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a-1},$$

$$\frac{\partial}{\partial\lambda}\ell(M;x)=\frac{\theta\lambda^{\theta-1}n}{\lambda^{\theta}}-\theta\lambda^{\theta-1}\sum_{i=1}^{n}x_i^{\theta}-\theta(\alpha-1)\lambda^{\theta-1}\sum_{i=1}^{n}\frac{x_i^{\theta}e^{-[\lambda x_i]^{\theta}}}{e^{-[\lambda x_i]^{\theta}}-1} \tag{25}$$

$$+\lambda^{\theta-1}\alpha\theta(\beta-1)\sum_{i=1}^{n}\frac{x^{\theta}e^{-[\lambda x_i]^{\theta}}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}}{\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}-1}$$

$$-\beta\alpha\theta(a-1)\lambda^{\theta-1}\sum_{i=1}^{n}\frac{x_i^{\theta}e^{-[\lambda x_i]^{\theta}}\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}}{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}-1}$$

$$+a\beta\alpha\theta(b-1)\lambda^{\theta-1}\sum_{i=1}^{n}\frac{x_i^{\theta}e^{-[\lambda x_i]^{\theta}}\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a-1}}{\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1},$$

and

$$\frac{\partial}{\partial\theta}\ell(M;x)=\sum_{i=1}^{n}\log(x_i)-\lambda^{\theta}\sum_{i=1}^{n}x_i^{\theta}\log[\lambda x_i]-(\alpha-1)\lambda^{\theta}\sum_{i=1}^{n}\frac{e^{-[\lambda x_i]^{\theta}}x_i^{\theta}\log[\lambda x_i]}{e^{-[\lambda x_i]^{\theta}}-1} \tag{26}$$

$$+\alpha(\beta-1)\lambda^{\theta}\sum_{i=1}^{n}\frac{e^{-[\lambda x_i]^{\theta}}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}x_i^{\theta}\log[\lambda x_i]}{\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}-1}+n\sum_{i=1}^{n}\frac{1+\theta\log\lambda}{\theta}$$

$$-\beta\alpha(a-1)\lambda^{\theta}\sum_{i=1}^{n}\frac{x^{\theta}e^{-[\lambda x_i]^{\theta}}\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}\log[\lambda x_i]}{\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}-1}$$

$$+a\beta\alpha(b-1)\lambda^{\theta}\sum_{i=1}^{n}\frac{x^{\theta}e^{-[\lambda x_i]^{\theta}}\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta-1}\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha-1}\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a-1}\log[\lambda x_i]}{\left[1-\left[1-\left[1-e^{-[\lambda x_i]^{\theta}}\right]^{\alpha}\right]^{\beta}\right]^{a}-1},$$

The MLEs of the unknown parameters are obtained by solving the nonlinear likelihood equations from (21) to (26) but They cannot be solved analytically, so we shall use a statistical software to solve the equations numerically. We can use iterative techniques such as a Newton–Raphson algorithm to obtain the estimate.

The Variance-Covariance Matrix

Let θ is the vector of the unknown parameters (*a, b, α, β, θ, λ*), The element of the 6 ×6 information matrix *I* (*a, b, α, β, θ, λ*) can be approximated by:

$$I_{ij}(\hat{\theta}) = -\frac{\partial^2 \ell(\theta)}{\partial\theta_i \, \partial\theta_j}\Big|_{\theta=\hat{\theta}}$$

and I^{-1}(*a, b, α, β, θ, λ*) is the variance-covariance matrix of the unknown parameters, where second derivatives can be easily driven via Mathcad, Maple, Matlab, or R.

The asymptotic distributions of the MLE parameters

$$\sqrt{n}\,(\hat{\theta}_i - \theta_i) \approx N_6(0, I^{-1}(\hat{\theta}_i)) \;\; , \; i = 1, ..., 6$$

The approximation 100 (1- γ) % confidence intervals of the unknown parameters based on the asymptotic distribution of the Kw Kw W (*a, b, α, β, θ, λ*) distribution are determined respectively as

$$\hat{\theta}_i \pm Z_{\frac{\gamma}{2}} \sqrt{I^{-1}(\hat{\theta}_i)} \; , i = 1, ..., 6$$

9. Illustration Study

An experiment is given to illustrate new results of the Kw Kw W (*a, b, α, β, θ, λ*) distribution, This study is about MLEs of parameters of the Kw Kw W distribution. The algorithm of obtaining parameters estimates is described in the following steps:

Step (1): Generate a random sample of size n as follows: $u_1, u_2, .., u_n$, by using the uniform distribution (0,1)

Step (2): transform the uniform random numbers to random numbers of the Kw Kw W distribution by using the quantile function of the Kw Kw W distribution:

Step (3): Solve the 21 to 26 by iteration to get the maximum likelihood estimators via iterative techniques and repeat it many times.

In this experiment 10, 15, 30 and 50 random numbers were generated using Mathcad package, then we obtained MLEs the Kw Kw W(*a, b, α, β, θ, λ*) distribution, where we started with parameters values: *a =2*, *b = 5*, α = *4*, *β = 3*, *θ=2* and *λ=3* for 1000 times, then we used the conjugate gradient iteration method. Finally, we get the following results

Sample size	Parameters	Biases	RMSEs
10	a	0.226	1.668
	b	-1.452	6.02
	α	1.449	2.291
	β	1.038	2.228
	θ	2.418	3.422
	λ	0.425	1.176
15	a	0.132	1.57
	b	-0.986	3.571
	α	1.268	2.14
	β	0.98	2.102
	θ	2.246	3.236
	λ	0.342	1.017
30	a	-0.074	1.269
	b	-0.49	3.204
	α	0.788	1.57
	β	0.572	1.621
	θ	1.413	2.345
	λ	0.257	0.994
50	a	-0.131	1.07
	b	-0.107	2.64
	α	0.533	1.339
	β	0.411	1.316
	θ	0.97	1.762
	λ	0.164	0.871

We see that the more sample size increases the more Biases and RMSEs decrease.

10. Application

In this section we give a real data to illustrate an example for one distribution of the new family of Kw Kw distributions so called Kw Kw W distribution to see how the new model works practically and we shall use the Mathcad package version 15 to do that. In our example, we used different distributions as the kumaraswamy - kumaraswamy - Weibull (Kw Kw W) distribution [derived from the Kw Kw family], the exponentiated kumaraswamy Weibull (E Kw W) distribution [derived from the E KW family], the kumaraswamy Weibull (Kw W) distribution [derived from the Kw family], the exponentiated generalized Weibull (EG W) distribution [derived from the E G family], the exponentiated Weibull (E-W) distribution [derived from the E family] and the

Weibull (W) distribution. The following data represents the lifetime (Hours) of T8 fluorescent lamps for 30 devices

0.228×10^5, 0.25×10^5, 0.256×10^5, 0.262×10^5, 0.263×10^5, 0.263×10^5, 0.264×10^5, 0.267×10^5, 0.272×10^5, 0.274×10^5, 0.277×10^5, 0.288×10^5, 0.288×10^5, 0.295×10^5, 0.297×10^5, 0.298×10^5, 0.302×10^5, 0.304×10^5, 0.305×10^5, 0.305×10^5, 0.31×10^5, 0.31×10^5, 0.311×10^5, 0.315×10^5, 0.325×10^5, 0.329×10^5, 0.349×10^5, 0.352×10^5, 0.359×10^5, 0.384×10^5

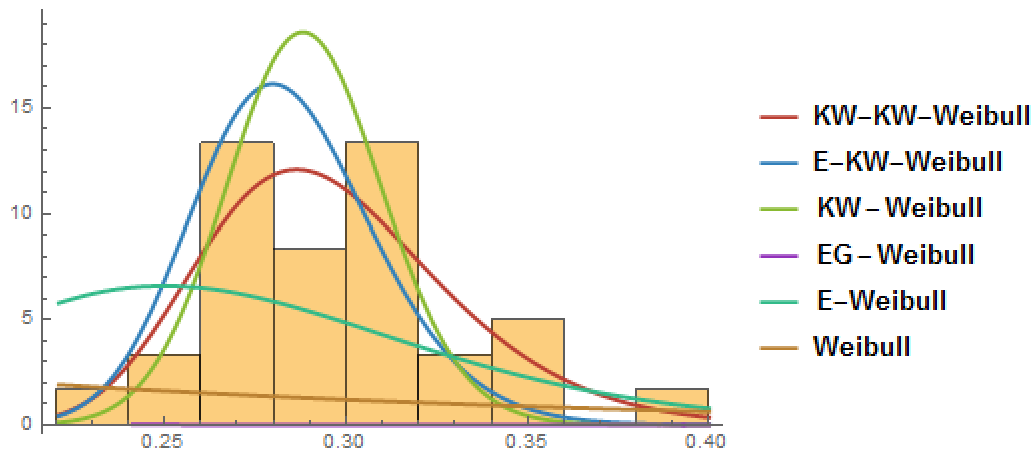

Figure 3. Probability density functions for different distributions

Table 1: The MLE of the parameter(s) and the associated AIC and BIC values

The Model	MLE of parameters						K.S	p-value	AIC	CAIC	BIC
	A	b	α	β	θ	Λ					
Kw Kw W (a, b, α, β, θ, λ)	4.998 (1.752)	0.482 (1.6)	5.116 (0.227)	4.55 (0.132)	2.431 (0.762)	4.537 (0.714)	0.087	0.023	-86.104	-82.452	-77.697
E Kw W (a, α, β, θ, λ)	5.01 (1.724)	1 –	5.366 (1.146)	4.296 (0.426)	2.432 (0.418)	4.506 (1.007)	0.254	0.958	-73.684	-71.184	-66.678
Kw W (α, β, θ, λ)	1 –	1 –	3.792 (3.285)	3.735 (1.864)	2.602 (1.479)	5.265 (1.129)	0.931	0.998	-59.802	-58.202	-54.197
EG W (a, β, θ, λ)	7.543 (3.355)	1 ()	1 ()	6.418 (0.747)	1.851 (1.671)	5.592 (0.222)	0.999	0.999	-69.588	-67.988	-63.983
E W (α, θ, λ)	1 –	1 –	7.201 (4.117)	1 –	1.949 (2.107)	5.987 (1.421)	0.406	0.999	-25.39	-24.467	-21.186
W(θ, λ)	1 –	1 –	1 –	1 –	1.163 (2.473)	5.297 (0.992)	0.717	0.999	-23.038	-22.594	-20.236

In the table (1) we compute the MLE of distributions parameters, the corresponding RMSE (given in parentheses), Kolmogorov-Smirnov (K.S) test statistic, AIC (Akaike Information Criterion), CAIC (the consistent Akaike Information Criterion) and BIC (Bayesian information criterion) for every distribution. We find from K.S test statistic that at the level of significant 0.01 we can not reject that the data fits all earlier distributions but it fits more the Kw Kw W (a, b, α, β, θ, λ) distribution. We see that the Kw Kw W (a, b, α, β, θ, λ) distribution has the smallest KS, AIC, CAIC and BIC so the Kw Kw W (a, b, α, β, θ, λ) distribution can be the best fitted distribution compared with other distributions.

Table 2: The log-likelihood function, The likelihood ratio test statistic and p-values

The Model	H_0	ℓ (log likelihood)	Λ (The likelihood ratio test statistic)	df (degrees of freedom)	P-value
E Kw W (a,α,β,θ,λ)	b=0	41.842	14.42	1	1.462×10^{-4}
Kw W (α, β, θ, λ)	a=0, b=0	33.901	30.302	2	2.63×10^{-7}
EG W (a, β, θ, λ)	α =0, b=0	38.794	20.516	2	3.508×10^{-5}
E W (α, θ, λ)	a=0, b=0, β=0	15.695	66.714	3	2.154×10^{-14}
W(θ, λ)	a=0, b=0,α=0,β=0	13.519	71.066	4	1.354×10^{-14}

*Note that the log likelihood of the Kw Kw W (a,b,α,β,θ,λ) = 49.052

In the table (2) and based on the likelihood ratio test, where the Kw Kw W (a, b, α, β, θ, λ) distribution generalizes the E Kw W (a, α, β, θ, λ) distribution, the Kw W (α, β, θ, λ) distribution, the EG W (a, β, θ, λ) distribution, the E W (α, θ, λ) distribution, the E W (α, θ, λ) distribution and the W(θ, λ) distribution, we find from the p-values that we can reject all null hypotheses when the level of significant is 0.01

11. Conclusions

The Kw Kw W distribution is an important distribution in world of data sets because of its flexible properties and its generalization of some important distributions as the E Kw W distribution, the Kw W distribution, the EG W distribution, the E W distribution and the W distribution. We encourage researchers to do more researches and applications on the Kw Kw W in univariate and multivariate cases.

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