Modified Maximum Likelihood Estimation from Censored Samples in Burr Type X Distribution

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Abstract

The two parameter Burr type X distribution is considered and its scale parameter is estimated from a censored sample using the classical maximum likelihood method. The estimating equations are modified to get simpler and efficient estimators. Two methods of modification are suggested. The small sample efficiencies are presented.

Keywords: MLE, Censored Samples, Order Statistics, Asymptotic Variance.

1. Introduction

In reliability studies exponential distribution is the central model for the study of any phenomenon through a probabilistic approach. Extension of this model into two different directions yields two popular models called the Weibull distribution and the gamma distribution. Between these two, the Weibull distribution is more frequently applied model in any practical situation concerning reliability studies. Both Weibull and gamma distributions involve a shape parameter, in the sense that the shape of the frequency curve of these models changes according as the change in the values of the shape parameters. In particular, for the Weibull distribution its shape parameter classifies it into an Increasing Failure Rate (IFR) model or a Decreasing Failure Rate (DFR) model or Constant Failure Rate (CFR) model (the well known exponential model) as its special cases. The natural phenomenon in reliability is “The aging concept” – indicated by increasing instantaneous failure probability with age of the product. A specific case of Weibull distribution exhibiting aging effect with an integer valued shape parameter is known as “The Rayleigh distribution”. Its cumulative distribution function (CDF), probability density function (PDF) and hazard function are given by

$$F(x) = 1 - e^{-x^2} \tag{1.1}$$

$$f(x) = 2xe^{-x^2} \tag{1.2}$$

$$h(x) = 2x \tag{1.3}$$

If F(x) is the cumulative distribution function of a positive valued random variable, then $[F(x)]^k$, k>0 also satisfies all the requirements for the cumulative distribution function of another positive valued random variable. It can be interpreted as the failure probability of a parallel system of k- components whose life times are independently and identically distributed random variables, each with a common CDF – F(x) if k is an integer. Exploring this concept to non integer values of k also, many researchers in the recent past made extensive studies on models of the type $[F(x)]^k$ generated by a basic well known model F(x). Such new models are named as exponentiated models by some authors and generalised models by some authors. For instance if the basic F(x) is exponential, $[F(x)]^k$ is named as generalised exponential (Gupta & Kundu- 1999), if F(x) is Weibull, $[F(x)]^k$ is named as exponentiated Weibull (Mudholkar and Srivastava – 1993). Banking on this notion, generalised Rayleigh distribution was studied by Raqab and Kundu (2006), as a process of revisit to Burr type X distribution. Its cumulative distribution function (CDF), probability density function (PDF) and hazard function are given by

$$F(x;k) = (1-e^{-x^2})^k;\ x>0,\ k>0 \tag{1.4}$$

$$f(x;k) = 2kxe^{-x^2}(1-e^{-x^2})^{k-1};\ \ x>0,\ k>0 \tag{1.5}$$

$$h(x;k) = \frac{2kxe^{-x^2}(1-e^{-x^2})^{(k-1)}}{1-(1-e^{-x^2})^k} \tag{1.6}$$

Burr (1942) has suggested a number of forms of cumulative distribution functions that might be useful in modeling various practical situations. In all, he suggested twelve models as listed below.

$$\left.\begin{array}{ll}
\textbf{(I)} & F(x) = x;\ \ 0<x<1, \\
\textbf{(II)} & F(x) = (e^{-x}+1)^{-k}, \\
\textbf{(III)} & F(x) = (x^{-c}+1)^{-k};\ \ 0<x<\infty, \\
\textbf{(IV)} & F(x) = \left[\left(\frac{c-x}{x}\right)^{1/c}+1\right]^{-k};\ \ \ 0<x<c, \\
\textbf{(V)} & F(x) = (ce^{-\tan x}+1)^{-k};\ -\frac{\pi}{2}<x<\frac{\pi}{2}, \\
\textbf{(VI)} & F(x) = \left(ce^{-k\sinh x}+1\right)^{-k}, \\
\textbf{(VII)} & F(x) = 2^{-k}(1+\tanh x)^k, \\
\textbf{(VIII)} & F(x) = \left(\frac{2}{\pi}\, arc\tan e^x\right)^k, \\
\textbf{(IX)} & F(x) = 1-\frac{2}{c[(1+e^x)^k-1]+2}, \\
\textbf{(X)} & F(x) = \left(1-e^{-x^2}\right)^k; 0<x<\infty, \\
\textbf{(XI)} & F(x) = \left(x-\frac{1}{2\pi}\sin 2\pi x\right)^k;\ \ 0<x<1, \\
\textbf{(XII)} & F(x) = 1-(1+x^c)^{-k}; 0<x<\infty.
\end{array}\right\} \tag{1.7}$$

Thus generalised Rayleigh distribution and Burr type X distributions are one and the same.

In the above models k and c are the positive parameters involved in the respective models. Among these twelve forms, the type X and type XII models are most frequently applied by many researchers. Our focus is Burr Type X model, whose expressions are given in equations (1.4), (1.5), and (1.6). If a scale parameter say λ is introduced, the cumulative distribution function, probability density function and hazard function are given by

$$F(x;k,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^k; \;\; x > 0, k > 0, \lambda > 0, \tag{1.8}$$

$$f(x;k,\lambda) = 2k\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{(k-1)}; x > 0, k > 0, \lambda > 0, \tag{1.9}$$

$$h(x;k,\lambda) = \frac{2k\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{(k-1)}}{1-\left(1-e^{-(\lambda x)^2}\right)^k}. \tag{1.10}$$

Expressions in (1.4), (1.5) and (1.6) are called standard Burr type X model, those in equations (1.8), (1.9) and (1.10) are called scaled Burr type X or Two parameter Burr type X model.

In this paper estimation of λ from type II censored sample where k is known is studied. Kantam and Ravikumar (2015) obtained modified maximum likelihood estimation (MMLE) of λ from complete samples. For a ready reference estimation from complete samples is presented in Section – 2. We discuss estimation from censored samples in Section - 3. This section includes the asymptotic as well as small sample behavior of the estimates. As these processes involve a lot of numerical computations, we present our results in the forms various numerical tables towards the end of respective sections of this paper with the respective labels of identification.

2. Estimation from Complete Samples

The probability density function of the two parameter Burr type X distribution is given by

$$f(x;k,\lambda) = 2k\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{(k-1)}; x > 0, k > 0, \lambda > 0 \tag{2.1}$$

Let $x_1<x_2<x_3<\ldots<x_n$ be a complete ordered sample of size n drawn from the above distribution (Though for complete samples maximum likelihood estimation does not require ordering of the sample, in order to facilitate presenting the methods to be introduced for this Section, which depend on ordered samples we consider ordered complete samples in the beginning itself, for uniformity in the notation.). The log likelihood equations to get the maximum likelihood estimates of λ and k are given by (after simplification).

$$\frac{\partial logL}{\partial \lambda} = 0 \Rightarrow \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 + 2\lambda(k-1)\sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{1-e^{-(\lambda x_i)^2}} = 0 \tag{2.2}$$

$$\frac{\partial logL}{\partial k} = 0 \Rightarrow \frac{n}{k} + \sum_{i=1}^{n} \ln\left(1 - e^{-(\lambda x_i)^2}\right) = 0 \tag{2.3}$$

These equations show that maximum likelihood estimator of λ is an iterative solution involving k and maximum likelihood estimator of k is a closed form expression involving λ given as

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n} \ln\left(1-e^{-(\lambda x_i)^2}\right)} \tag{2.4}$$

Since λ is a scale parameter, in view of the scale invariant nature of the Burr type X distribution, maximum likelihood estimation of *k* can be taken as the following expression in a standard model.

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n} \ln\left(1-e^{-z_i^2}\right)} \tag{2.5}$$

where $\lambda x_i = z_i$.

The elements of the information matrix and hence those of the asymptotic dispersion matrix require the following second order partial derivatives.

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda^2} = \frac{-2}{\lambda^2} - 2x^2 + \frac{2x^2(k-1)e^{-(\lambda x)^2}(1-2\lambda^2 x^2 - e^{-(\lambda x)^2})}{[1-e^{-(\lambda x)^2}]^2} \tag{2.6}$$

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial k^2} = \frac{-1}{k^2} \tag{2.7}$$

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k} = \frac{2x^2\lambda e^{-(\lambda x)^2}}{1-e^{-(\lambda x)^2}} \tag{2.8}$$

The information matrix is given by

$$\begin{bmatrix} -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda^2}\right] & -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k}\right] \\ -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k}\right] & -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial k^2}\right] \end{bmatrix}$$

As the integrals involved in these mathematical expectations are not analytically tractable we have evaluated them using 11- point Gauss-Laguerre quadrature formula (Rao *et al.*, 1966) for selected values of *k* in a standard density (λ=1) and are given in Table - 2.1. The elements of the corresponding asymptotic dispersion matrix for selected values of *k* are given in Table – 2.2.

Table 2.1: Elements to get Information Matrix at *k*=2 and 3

k	**2**			**3**		
Matrix	**I_{11}**	**I_{12}**	**I_{22}**	**I_{11}**	**I_{12}**	**I_{22}**
	8.84668	-0.989225	0.25	10.9596	-0.68296	0.1111

Table 2.2: Elements of Asymptotic Dispersion Matrix of MLEs of λ & *k*

k / ***n***	**2**			**3**		
	σ_{11}	σ_{12}	σ_{22}	σ_{11}	σ_{12}	σ_{22}
5	0.040548	0.160445	1.434864	0.029579	0.181811	2.917553
10	0.020274	0.080222	0.717432	0.014789	0.090905	1.458777
15	0.013516	0.053482	0.478288	0.00986	0.060604	0.972518
20	0.010137	0.040111	0.358716	0.007395	0.045453	0.729388
25	0.00811	0.032089	0.286973	0.005916	0.036362	0.583511

When the log likelihood equations do not admit analytical expressions as MLEs of the parameters of a density function from complete or censored sample, replacement of certain portions of log likelihood equations by suitable admissible approximations sometimes would lead to simpler and efficient estimates of the parameters. Such estimates in literature are named as approximate or modified MLEs. Tiku (1967); Mehrotra and Nanda (1974); Pearson and Rootzen (1977); Tiku and Suresh (1992); Rosaiah *et al.* (1993a); Rosaiah *et al.* (1993b); Kantam and Srinivasa Rao (1993); Kantam and Srinivasa Rao (2002); Kantam and Sriram (2003); Kantam *et al*. (2013) and the references therein are some of the works in this direction. We adopt this concept of MML estimation for Burr type X distribution by considering its reparameterised version as given in Raqab and Kundu (2006).

We see that the maximum likelihood estimator of the shape parameter k is a closed form expression whereas, the maximum likelihood estimator of λ is an iterative solution of the equation (2.2) involving k. In order to overcome the iterative nature of the solution we proceed as follows. Equation (2.2) can be rewritten as

$$2n - 2\sum_{i=1}^{n} z_i^2 + 2(k-1)\sum_{i=1}^{n} \frac{z_i^2 e^{-z_i^2}}{1-e^{-z_i^2}} = 0, \tag{2.9}$$

where $z_i = \lambda x_i$.

Consider the expression $g(z_i) = \frac{z_i^2 e^{-z_i^2}}{1-e^{-z_i^2}}$ (2.10)

of equation (2.9).

We approximate this expression by a linear one in z_i in a small neighborhood of the i^{th} quantile of the population say

$$g(z_i) \cong \alpha_i + \beta_i z_i. \tag{2.11}$$

With this approximation, equation (2.9) becomes a quadratic equation in λ given by

$$A\lambda^2 + B\lambda + C = 0. \tag{2.12}$$

where, $A = \sum_{i=1}^{n} z_i^2$, $B = -(k-1)\sum_{i=1}^{n} \beta_i z_i$, $C = -n - (k-1)\sum_{i=1}^{n} \alpha_i$.

Positive root of this equation is an estimate of λ called the modified maximum likelihood estimate (MMLE) of λ. It can be seen that A,B,C depend on the ordered observations $x_1, x_2, \dots, x_n$, the shape parameter k and the slope, intercept of the linear approximation (2.11). In order to find α_i, β_i, we suggest two methods.

Method-I:

Let $p_i = \frac{i}{n+1}, i = 1,2,..n$.

Let z_i^*, z_i^{**} be the solutions of the following equations

$F(z_i^*) = p_i^*, F(z_i^{**}) = p_i^{**}$,

where $p_i^* = p_i - \sqrt{\frac{p_i q_i}{n}}, p_i^{**} = p_i + \sqrt{\frac{p_i q_i}{n}}$, F(.) is the cdf of standard Burr type X distribution, and $q_i = 1 - p_i$.

The expressions for z_i^*, z_i^{**} are

$$z_i^* = \sqrt{-Ln\left[1-\left(p_i - \sqrt{\frac{p_i q_i}{n}}\right)^k\right]}\,, \tag{2.13}$$

$$z_i^{**} = \sqrt{-Ln\left[1-\left(p_i + \sqrt{\frac{p_i q_i}{n}}\right)^k\right]} \tag{2.14}$$

The slope β_i and intercept α_i of the linear approximation in the equation (2.11) are given by

$$\beta_i = \frac{g(Z_i^{**})-g(Z_i^*)}{Z_i^{**}-Z_i^*}, \tag{2.15}$$

$$\alpha_i = g(z_i^*) - \beta_i z_i^*. \tag{2.16}$$

where β_i is given by (2.15). The values of α_i and β_i in this method for n=5(5)25 for k=0.25, 0.50, 1.50, 2, 2.5,3 are given in Table 2.3 in Appendix-I.

Method-II:

Considering Taylor's expansion of $g(zi)$ upto its first derivative w.r.t z_i in a neighborhood of population quantile corresponding to p_i, we get

$$\beta_i = g'(z_i), \tag{2.17}$$

$$\alpha_i = g(z_i) - \beta_i z_i, \tag{2.18}$$

where z_i is the quantile of Burr type X distribution, given as the solution of the equation $F(z_i) = p_i$.

i.e., $$z_i = \sqrt{-Ln\left[1-\left(p_i - \sqrt{\frac{p_i q_i}{n}}\right)^k\right]} \tag{2.19}$$

It can be seen from (2.10) that

$$g'(z_i) = \frac{2z_i e^{-z_i^2}(1-z_i-e^{-z_i^2}+z_i e^{-z_i^2}-z_i^2 e^{-z_i^2})}{(1-e^{-z_i^2})^2} \tag{2.20}$$

Substituting (2.19) in (2.20) we get

$$\beta_i = g'(z_i) = \frac{2z_i e^{-z_i^2}(1-z_i-e^{-z_i^2}+z_i e^{-z_i^2}-z_i^2 e^{-z_i^2})}{(1-e^{-z_i^2})^2} \tag{2.21}$$

Using β_i in (2.18) we get α_i.

The values of α_i and β_i in this method for n=5(5)25 for k=0.25, 0.50, 1.5, 2, 2.5,3 are given in Table 2.4 in Appendix-II.

In the above two modified methods, the basic principle is that certain expressions in the log likelihood equation are linearised in a neighborhood of the population quantile which depends on the size of the sample also. The larger the size, the narrower is the neighborhood and hence the closer is the approximation. That is, the exactness of the approximation becomes finer and finer for large values of 'n'. Hence the approximate log likelihood equation and the exact log likelihood equation tend to each other as $n \to \infty$. Hence the exact and modified MLEs are asymptotically identical (Tiku *et al.* 1986). The same may not be true in small samples and these are to be assessed with the help of small sample characteristics of the MMLEs. Because of non-*tract*ability of analytical sampling variances, we compared the modified ML method with exact ML method through Monte–Carlo simulation.

10,000 random samples of size n= 5 (5) 25 each are generated from Burr type X distribution with k=0.25, 0.50, 1.5, 2, 2.5,3 in succession. For each sample at a given k the α_i and β_i of Method-I (Method-II) as given in Table 2.3 (2.4) are used in equation 2.12 to get the modified MLE of 'λ' by Method-I (Method-II). The empirical variances of MMLEs by Method-I and Method-II are respectively given in Table 2.5 in Appendix - III.

3. Estimation from Censored Samples

In life testing experiments censoring a given sample sometimes becomes necessary to save time and cost of experimentation. One common scheme of censoring is a failure censored sample or a type II right censored sample, wherein pre-planned n items are put to life testing and the experiment will be terminated as soon as a prefixed observations say 'r' are noted down (r < n). In such situations, we are left with 'r' actual observations say $x_1 < x_2 < \cdots < x_r$ and the life times of the remaining (n-r) items are more than x_r. Such a sample is called type – II right censored sample.

In this Section, we assume that we have a type – II right censored sample modeled by Burr type X distribution with shape parameter 'k' and scale parameter 'λ'. Maximum likelihood estimation of scale parameter for a known shape parameter will be discussed below.

Let $x_1 < x_2 < \cdots < x_r$ be a type – II right censored sample from a Burr type X distribution in a planned random sample of sized 'n'. The likelihood function of such a censored sample is given by α is

$$\prod_{i=1}^{r} f(x_i;\lambda,k).[1-F(x_r;\lambda,k)]^{n-r} \tag{3.1}$$

where $f(\cdot), F(\cdot)$ respectively denote the probability density function and cumulative distribution function of Burr type X distribution. Substituting the respective expressions for f, F, taking natural logarithms, differentiating with respect to 'λ' and on simplification we get the estimating equation for the parameter 'λ' as

$$r - \sum_{i=1}^{r} z_i^2 + (k-1)\sum_{i=1}^{r} \frac{z_i^2 e^{-z_i^2}}{(1-e^{-z_i^2})} + \frac{k^2(n-r)z_r^2 e^{-z_r^2}(1-e^{-z_r^2})^{(k-1)}}{1-(1-e^{-z_r^2})^k} = 0, \tag{3.2}$$

where $z_i = \lambda x_i$

It can be seen that for known value of k, the MLE of λ from censored sample is an iterative solution of equation (3.2). However, an analytical solution can be obtained by admissible modifications to some terms of equation (3.2) on lines of the procedures described in Section – 2. Consider the following two expressions of equation (3.2).

$$g(z_i) = \frac{z_i^2 e^{-z_i^2}}{1-e^{-z_i^2}} \tag{3.3}$$

$$h(z_r) = \frac{z_r^2 e^{-z_r^2}(1-e^{-z_r^2})^{(k-1)}}{1-(1-e^{-z_r^2})^k} \tag{3.4}$$

We suggest to approximate $g(z_i)$ of (4.3), $h(z_r)$ of (3.4) by the following linear expressions in small neighborhoods of i^{th} population quantile, r^{th} population quantile respectively.

i.e., $g(z_i) \cong \alpha_i + \beta_i z_i$ (3.5)

$$h(z_r) \cong \Upsilon_r + \delta_r z_r \tag{3.6}$$

Substituting these approximations in equation (3.2) and on simplification we get the following quadratic equation in 'λ'.

$$A\lambda^2 + B\lambda + C = 0. \tag{3.7}$$

where $A = -\sum_{i=1}^{r} x_i^2$ (3.8)

$$B = (k-1)\sum_{i=1}^{r} x_i\beta_i + k^2(n-r)\delta_r x_r \tag{3.9}$$

$$C = r + (k-1)\sum_{i=1}^{r} \alpha_i + k^2(n-r)\Upsilon_r \tag{3.10}$$

Positive root of the above quadratic equation in (3.7) is taken as an estimate of λ called MMLE of λ from censored samples. We attempt to support our linear approximations for $h(z_r)$ in the following way.

Let $p = \frac{r+1}{n+1}$; $q = 1 - p$

$p^* = p - \sqrt{\frac{pq}{n}}$; $p^{**} = p + \sqrt{\frac{pq}{n}}$

z^* be the solution of $F(x) = p^*$

z^{**} be the solution of $F(x) = p^{**}$

The interval (z^*, z^{**}) is evenly divided by 10 cut off points say $z^*=z_1<z_2<\ldots<z_{10}=z^{**}$. The Karl-Pearson's product moment correlation coefficient between $(z_i, h(z_i))$, i=1, 2, 3… 10 is calculated for n=5(5) 25 with all possible values of 'r'. These are given in Table – 3.1 in Appendix - III, which show that there is a high linear relation between z and $h(z)$ indicating an acceptable linear approximation for $h(z)$ in the neighborhood z_r as expressed in the equation

$$h(z_r) \cong \Upsilon_r + \delta_r z_r. \tag{3.11}$$

It can be seen that the constants A,B,C of the above quadratic equation in (3.7) depend on the values of the uncensored observations, size of the planned sample, number of available observations, the known shape parameter k, the slopes and intercepts of the linear approximations for $g(z_i)$ of (3.5) and $h(z_r)$ of (3.6). We suggest as in Section – 2, two methods of finding these slopes and intercepts. It may also be noted that the slope (β_i), the intercept (α_i) in the linear approximation for $g(z_i)$ remain unaltered from those of Section - 2. The slope (δ_r) and the intercept (Υ_r) of the linear approximation of $h(z_r)$ will now be evaluated by methods – I and II of Section – 2.

Method – I:

Let $p_r = \frac{r}{n+1}$, $q_r = 1 - p_r$.

$$p_r' = p_r - \sqrt{\frac{p_r q_r}{n}}\ , \qquad p_r'' = p_r + \sqrt{\frac{p_r q_r}{n}}$$

Let ξ_r', ξ_r'' be the respective solutions of $F(x_r) = p_r', F(x_r) = p_r''$,
where F(.) is the CDF of Burr type X distribution.

i.e., $\xi_r' = \sqrt{-Ln(1-(p_r')^{1/k})}$ (3.12)

$$\xi_r'' = \sqrt{-Ln(1-(p_r'')^{1/k})} \tag{3.13}$$

The slope $\delta_r = \frac{h(\xi_r'') - h(\xi_r')}{\xi_r'' - \xi_r'}$ (3.14)

and intercept $\Upsilon_r = h(\xi_r'') - \delta_r \xi_r''$ (3.15)

The values of Υ_r, δ_r for n=5(5)25, for all possible values of r within each 'n' are tabulated in Tables 3.2 to 3.7 in Appendix-IV.

Method – II:

Expanding $h(z_r)$ in Taylor series up to the first derivative in the neighborhood of the r^{th} quantile of the population and after some simplification we get

$$\delta_r = h'(\xi_r) \tag{3.16}$$

From equation (3.4) we get that

$$h'(z_r) = \frac{2ze^{-z^2}\left(1-e^{-z^2}\right)^{k-1}\left[\left(1-\left(1-e^{-z^2}\right)^{k}\right)\left\{(k-1)z^2e^{-z^2}\left(1-e^{-z^2}\right)^{-1}+\left(1-z^2\right)\right\}+\left\{kz^2e^{-z^2}\left(1-e^{-z^2}\right)^{k-1}\right\}\right]}{\left[1-\left(1-e^{-z^2}\right)^{k}\right]^2} \tag{3.17}$$

Replacing z with ξ_r in equation (3.17) we get

$$\delta_r = h'(\xi_r) = \frac{2\xi_r e^{-\xi_r^2}\left(1-e^{-\xi_r^2}\right)^{k-1}\left[\left(1-\left(1-e^{-\xi_r^2}\right)^{k}\right)\left\{(k-1)\xi_r^2e^{-\xi_r^2}\left(1-e^{-\xi_r^2}\right)^{-1}+\left(1-\xi_r^2\right)\right\}+\left\{k\xi_r^2e^{-\xi_r^2}\left(1-e^{-\xi_r^2}\right)^{k-1}\right\}\right]}{\left[1-\left(1-e^{-\xi_r^2}\right)^{k}\right]^2} \tag{3.18}$$

Let $F(x) = p = \frac{r}{n+1}$.

$$\text{Then } \xi_r = z = \sqrt{-Ln\left[1-\left(\frac{r}{n+1}\right)^{1/k}\right]} \tag{3.19}$$

$$\Upsilon_r = h(\xi_r) - \delta_r\xi_r \tag{3.20}$$

The values of Υ_r, δ_r for *n*=5(5)25 and all possible values of ***r*** are given in Tables 3.8 to 3.13 in Appendix-V. From Bhattacharyya (1985) it follows that the MMLEs from censored samples are also asymptotically as efficient as MLEs.

In small samples the performance of MMLEs between the two methods of modification is studied through Monte-Carlo simulation for selected values of *n*, *r* (to the extent of 20% censoring in each chosen value of *n*). That is, the selected combinations are *n*=5,*r*=4; *n*=10,*r*=8; *n*=15,*r*=12; *n*=20,*r*=16; *n*=25,*r*=20. These are given in Table 3.14 respectively for methods - I and II. The variances of MMLEs from censored samples are found to be more than the corresponding variances of MMLEs from complete sample (a natural trend) as evidenced from the comparison between Table (2.5, 3.14) in Appendix - VI.

Appendix - I

Table 2.3: Slope (β_i) and Intercept (α_i) in Modification to MLE by Method –I for n=5, 10, 15, 20

$n = 5$			$n = 10$						$n = 15$					
k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
0.25	1.001544	-0.06947	**0.25**	1.000137	-0.02066	**2.0**	1.173059	-0.57054	**0.25**	1.000031	-0.00977	**1.50**	1.098443	-0.43888
	1.002237	-0.15653		1.000165	-0.04796		1.176377	-0.58546		1.000035	-0.02292		1.100832	-0.45901
	1.020322	-0.30384		1.001491	-0.09438		1.220816	-0.63942		1.000313	-0.04533		1.143291	-0.53186
	1.080881	-0.48925		1.005975	-0.15585		1.245337	-0.66498		1.001251	-0.07506		1.177126	-0.58202
	1.150719	-0.54725		1.016637	-0.23229		1.252992	-0.67115		1.003479	-0.11212		1.204841	-0.61856
0.50	1.027824	-0.2493		1.037497	-0.32307		1.241769	-0.65958		1.007835	-0.1565		1.226972	-0.6449
	1.033775	-0.33597		1.073303	-0.42602		1.205548	-0.62816		1.015373	-0.20811		1.24326	-0.66247
	1.100058	-0.49402		1.127991	-0.53443		1.131001	-0.57027		1.027354	-0.26679		1.252849	-0.67167
	1.185074	-0.61542		1.195104	-0.62423		0.984942	-0.46837		1.045219	-0.3321		1.254213	-0.67217
	1.24172	-0.66128		1.241632	-0.66093		0.935784	-0.44881		1.070507	-0.40319		1.244874	-0.66286
1.50	1.182883	-0.58678	**0.50**	1.00827	-0.13626	**2.5**	1.206809	-0.61719		1.104591	-0.47831		1.220775	-0.64161
	1.190747	-0.61196		1.009135	-0.18243		1.208629	-0.62588		1.147842	-0.55364		1.174896	-0.60461
	1.239253	-0.65949		1.027432	-0.27312		1.242211	-0.6619		1.196673	-0.62001		1.093706	-0.54452
	1.207388	-0.62266		1.054795	-0.36228		1.253638	-0.67203		1.229891	-0.6498		0.944811	-0.44406
	1.182908	-0.61317		1.090811	-0.44839		1.24682	-0.66458		1.255411	-0.67339		0.901456	-0.42827
2.0	1.223006	-0.63852		1.134264	-0.52887		1.220169	-0.641	**0.50**	1.003907	-0.09372	**2.0**	1.145468	-0.52655
	1.22516	-0.64773		1.181872	-0.5988		1.168027	-0.59939		1.004177	-0.12522		1.147764	-0.5405
	1.24143	-0.65818		1.224414	-0.64754		1.077988	-0.53385		1.012541	-0.18772		1.189346	-0.59877
	1.166494	-0.58988		1.230982	-0.64614		0.91947	-0.42895		1.02509	-0.24989		1.218043	-0.63439
	1.122837	-0.56967		1.240338	-0.65936		0.866911	-0.4078		1.041799	-0.31145		1.238066	-0.65684
2.5	1.245223	-0.66349	**1.50**	1.125535	-0.49283	**3.0**	1.229802	-0.64553		1.062591	-0.37204		1.250599	-0.66957
	1.24134	-0.66169		1.129668	-0.51631		1.229873	-0.64953		1.087301	-0.43115		1.255704	-0.674
	1.229046	-0.64573		1.181483	-0.59155		1.251994	-0.67091		1.115588	-0.48807		1.252718	-0.67057
	1.122945	-0.55821		1.218877	-0.63751		1.25151	-0.66926		1.14679	-0.54168		1.240259	-0.65901
	1.066949	-0.5316		1.243112	-0.66304		1.232594	-0.65192		1.179624	-0.59023		1.216007	-0.63835
3.0	1.255781	-0.67419		1.251484	-0.66968		1.193878	-0.62009		1.211546	-0.63084		1.176159	-0.60677
	1.246254	-0.66445		1.237361	-0.65494		1.130075	-0.57196		1.237222	-0.65824		1.114203	-0.56097
	1.20989	-0.62921		1.186527	-0.61116		1.029602	-0.50217		1.244227	-0.66175		1.017789	-0.49469
	1.080905	-0.52942		1.062996	-0.51793		0.864134	-0.39689		1.196747	-0.61404		0.857917	-0.39339
	1.016665	-0.49883		1.020774	-0.50184		0.809966	-0.37503		1.187022	-0.61676		0.811801	-0.37616

$n = 15$						$n = 20$					
k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
2.50	1.181879	-0.58276	**3.0**	1.20889	-0.61947	**0.25**	1.00001	-0.00567	**0.50**	1.002268	-0.07141
	1.183569	-0.5921		1.209719	-0.62528		1.000011	-0.01338		1.002384	-0.09534
	1.219838	-0.63659		1.238976	-0.65781		1.000102	-0.02653		1.007157	-0.14297
	1.241245	-0.66013		1.252773	-0.67156		1.000408	-0.044		1.01432	-0.19049
	1.252889	-0.67168		1.256538	-0.67465		1.001135	-0.06579		1.02387	-0.23778
	1.256284	-0.67441		1.251899	-0.6699		1.002554	-0.0919		1.035796	-0.28475
	1.251662	-0.66962		1.239161	-0.65844		1.005008	-0.12232		1.050074	-0.33126
	1.238482	-0.65769		1.217848	-0.64057		1.008909	-0.15703		1.066657	-0.37713
	1.215483	-0.63836		1.186791	-0.61602		1.014735	-0.196		1.085462	-0.42213
	1.180508	-0.61078		1.143973	-0.584		1.023035	-0.23913		1.106348	-0.46596
	1.130019	-0.5733		1.086092	-0.54302		1.034416	-0.28628		1.129082	-0.50822
	1.058	-0.52303		1.007571	-0.49048		1.049536	-0.33721		1.153283	-0.54832
	0.953159	-0.45451		0.898018	-0.42152		1.069067	-0.3915		1.17832	-0.58547
	0.789276	-0.35511		0.733478	-0.32497		1.093623	-0.44843		1.203138	-0.61845
	0.742451	-0.33748		0.686858	-0.30739		1.123581	-0.50675		1.22592	-0.64544
							1.15865	-0.56415		1.243396	-0.66347
							1.196691	-0.6161		1.249247	-0.66734
							1.230041	-0.65247		1.229782	-0.64682
							1.230159	-0.64503		1.148085	-0.57665
							1.241405	-0.66039		1.128606	-0.57428

Appendix – I (Continued)

Table 2.3 (Continued): Slope (β_i) and Intercept (α_i) in Modification to MLE by Method –I for n=20, 25

$n = 20$

k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
1.50	1.082349	-0.40267	**2.0**	1.12777	-0.49543	**2.5**	1.164819	-0.55714	**3.0**	1.193592	-0.59892
	1.083927	-0.42055		1.129445	-0.50833		1.166204	-0.56626		1.194464	-0.60511
	1.119918	-0.48957		1.167338	-0.56656		1.201462	-0.6132		1.22501	-0.64179
	1.149383	-0.53876		1.194992	-0.60454		1.224602	-0.64114		1.242742	-0.66125
	1.17465	-0.57661		1.216345	-0.63134		1.240467	-0.65881		1.252899	-0.67154
	1.19651	-0.60651		1.232809	-0.65047		1.250798	-0.66949		1.257349	-0.67562
	1.215238	-0.63015		1.244968	-0.66362		1.256342	-0.67476		1.256901	-0.67489
	1.230836	-0.64844		1.253007	-0.67169		1.257376	-0.67543		1.251869	-0.67007
	1.243116	-0.66184		1.256848	-0.67515		1.253888	-0.67191		1.242269	-0.66151
	1.251708	-0.67052		1.256201	-0.67417		1.245636	-0.66437		1.22789	-0.64934
	1.256041	-0.67442		1.250557	-0.6687		1.232164	-0.65275		1.208307	-0.63349
	1.255289	-0.67322		1.239151	-0.6585		1.21276	-0.63681		1.182859	-0.61375
	1.248272	-0.6664		1.220875	-0.64308		1.186396	-0.61614		1.150583	-0.58972
	1.233288	-0.65306		1.194131	-0.62168		1.151585	-0.59006		1.110092	-0.56078
	1.207801	-0.63184		1.156561	-0.59313		1.106143	-0.55752		1.05936	-0.52601
	1.167858	-0.60063		1.104525	-0.5556		1.046727	-0.51693		0.9953	-0.48398
	1.106833	-0.55591		1.03201	-0.50613		0.96786	-0.4657		0.912886	-0.43238
	1.012267	-0.49121		0.927889	-0.43927		0.859507	-0.39914		0.802972	-0.36708
	0.854987	-0.39175		0.766522	-0.34273		0.698763	-0.30665		0.644812	-0.27875
	0.812791	-0.37677		0.723591	-0.32722		0.656338	-0.29125		0.60336	-0.26369

$n = 25$

k	α_i	β_i	k	α_i	β_i
0.25	1.000004	-0.0037	**0.50**	1.001479	-0.05769
	1.000005	-0.00876		1.00154	-0.07698
	1.000043	-0.0174		1.004622	-0.11545
	1.00017	-0.02888		1.009246	-0.15387
	1.000474	-0.04321		1.015413	-0.19218
	1.001066	-0.06037		1.023121	-0.23035
	1.002089	-0.08039		1.032364	-0.26832
	1.003715	-0.10324		1.043131	-0.30601
	1.006144	-0.12893		1.055404	-0.34335
	1.009603	-0.15743		1.069153	-0.38025
	1.01435	-0.18872		1.08433	-0.41658
	1.020671	-0.22276		1.100865	-0.45218
	1.028875	-0.25948		1.11865	-0.48687
	1.039297	-0.29877		1.137526	-0.52038
	1.052286	-0.34049		1.157255	-0.5524
	1.068191	-0.3844		1.177482	-0.58246
	1.087336	-0.43013		1.197669	-0.60997
	1.109964	-0.47713		1.216991	-0.63407
	1.136123	-0.5245		1.234142	-0.65355
	1.165428	-0.57076		1.24699	-0.66655
	1.19652	-0.61331		1.25187	-0.67014
	1.225693	-0.64725		1.241994	-0.65932
	1.242853	-0.66217		1.203239	-0.62441
	1.215516	-0.63019		1.098987	-0.54209
	1.216785	-0.63998		1.073867	-0.53682

$n = 25$

k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
1.50	1.071525	-0.37604	**2.0**	1.115236	-0.47176	**2.50**	1.152241	-0.53706	**3.0**	1.18188	-0.58232
	1.072659	-0.39233		1.116526	-0.48382		1.153386	-0.54585		1.182687	-0.58855
	1.104095	-0.45789		1.151321	-0.54107		1.187112	-0.59352		1.213117	-0.62714
	1.130135	-0.50536		1.177333	-0.5795		1.210229	-0.62331		1.232173	-0.64938
	1.152864	-0.5427		1.198176	-0.60779		1.227257	-0.64368		1.244813	-0.66315
	1.173059	-0.57314		1.215216	-0.62932		1.239895	-0.65786		1.25288	-0.6714
	1.191077	-0.59835		1.229108	-0.64581		1.24896	-0.66746		1.257256	-0.67558
	1.207077	-0.61929		1.240186	-0.65822		1.254877	-0.67336		1.258402	-0.67648
	1.2211	-0.63657		1.248602	-0.66713		1.257854	-0.67609		1.256548	-0.67457
	1.2331	-0.65052		1.254384	-0.67291		1.25796	-0.67599		1.251781	-0.67012
	1.242959	-0.66135		1.257466	-0.67575		1.25516	-0.67321		1.244078	-0.6633
	1.250485	-0.66916		1.257695	-0.67573		1.249328	-0.66785		1.23333	-0.65415
	1.255405	-0.67392		1.254833	-0.67286		1.240249	-0.6599		1.219339	-0.64267
	1.257351	-0.67553		1.248542	-0.66705		1.227613	-0.64927		1.201816	-0.62877
	1.255834	-0.67379		1.238364	-0.65812		1.210996	-0.63581		1.180364	-0.61231
	1.250205	-0.6684		1.223689	-0.64581		1.189827	-0.61927		1.154451	-0.59307
	1.239589	-0.6589		1.203697	-0.62972		1.163345	-0.59931		1.123366	-0.57073
	1.222792	-0.64466		1.177273	-0.60932		1.130518	-0.57545		1.086151	-0.54486
	1.198137	-0.62479		1.142868	-0.58382		1.089915	-0.54701		1.041482	-0.51485
	1.163179	-0.59798		1.098242	-0.55212		1.03949	-0.51301		0.987474	-0.47983
	1.114166	-0.56227		1.040004	-0.51257		0.976159	-0.47203		0.921303	-0.43853
	1.044902	-0.5145		0.962625	-0.46249		0.89494	-0.42175		0.838434	-0.38893
	0.943902	-0.44892		0.855998	-0.39713		0.786808	-0.35809		0.730694	-0.32743
	0.784629	-0.35259		0.697097	-0.30579		0.631357	-0.27191		0.579713	-0.24602
	0.74403	-0.33841		0.656918	-0.29157		0.592338	-0.25806		0.542055	-0.23266

Appendix – II

Table 2.4: Slope (β_i) and Intercept (α_i) in Modification to MLE by Method –II for n=5, 10, 15, 20

$n=5$			$n=10$						$n=15$					
k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
0.25	1.249164	-0.66728	**0.25**	1.369003	-0.80108	**2.0**	1.169044	-1.90112	**0.25**	1.421925	-0.87535	**1.50**	1.225577	-1.86011
	1.032638	-0.48712		1.227534	-0.64647		1.310984	-1.7873		1.312165	-0.73308		1.349387	-1.74887
	0.837227	-0.36099		1.107134	-0.543		1.423283	-1.65974		1.219567	-0.639		1.435953	-1.64161
	0.638381	-0.25422		0.99642	-0.46171		1.502926	-1.51928		1.136041	-0.56619		1.495812	-1.53547
	0.408803	-0.14952		0.889993	-0.39261		1.546091	-1.3664		1.057889	-0.50548		1.533399	-1.42939
0.50	1.497945	-1.01809		0.784217	-0.33074		1.547629	-1.20121		0.982965	-0.45254		1.550882	-1.32281
	1.33462	-0.75879		0.675774	-0.27299		1.50011	-1.02333		0.9098	-0.4049		1.54924	-1.21534
	1.136041	-0.56619		0.560626	-0.21685		1.391787	-0.83146		0.837227	-0.36099		1.528641	-1.10668
	0.901726	-0.39986		0.432266	-0.15948		1.201172	-0.62214		0.764192	-0.31967		1.488557	-0.99648
	0.603306	-0.2371		0.275998	-0.09586		0.876108	-0.38413		0.6896	-0.28007		1.427703	-0.88435
1.50	1.410392	-1.67716	**0.50**	1.546728	-1.19433	**2.5**	1.096036	-1.94764		0.612156	-0.24138		1.343793	-0.7697
	1.541392	-1.39394		1.48529	-0.98934		1.215465	-1.86781		0.530129	-0.20276		1.232961	-0.65162
	1.528641	-1.10668		1.399023	-0.84158		1.331993	-1.76681		0.440875	-0.16318		1.088414	-0.52846
	1.374521	-0.80824		1.300797	-0.72057		1.433348	-1.64543		0.339641	-0.12101		0.896671	-0.39672
	1.030754	-0.48577		1.19328	-0.61506		1.509276	-1.50373		0.215309	-0.07281		0.622995	-0.24665
2.0	1.289313	-1.80725		1.07626	-0.5192		1.549232	-1.34139	**0.50**	1.552458	-1.28535	**2.0**	1.119018	-1.93369
	1.480221	-1.56749		0.947751	-0.42915		1.540639	-1.1576		1.528641	-1.10668		1.225577	-1.86011
	1.552458	-1.28535		0.803547	-0.3416		1.46619	-0.95071		1.480384	-0.97894		1.318899	-1.77972
	1.471532	-0.96104		0.63494	-0.25252		1.29738	-0.71686		1.421925	-0.87535		1.398141	-1.69287
	1.158678	-0.58502		0.419563	-0.15407		0.970534	-0.44418		1.357266	-0.7862		1.462347	-1.59984
2.5	1.195363	-1.88263	**1.50**	1.287698	-1.80869	**3.0**	1.053683	-1.97185		1.287851	-0.70668		1.510406	-1.50084
	1.401892	-1.68814		1.429281	-1.65129		1.145805	-1.91664		1.214131	-0.63397		1.540989	-1.396
	1.534466	-1.42518		1.51191	-1.49693		1.252964	-1.83836		1.136041	-0.56619		1.552458	-1.28535
	1.524217	-1.09133		1.549142	-1.34224		1.36101	-1.73634		1.053129	-0.50198		1.542733	-1.16885
	1.25667	-0.6747		1.545511	-1.18584		1.456822	-1.6091		0.964558	-0.4402		1.509074	-1.04629
3.0	1.12898	-1.92745		1.50148	-1.02671		1.526279	-1.45485		0.868973	-0.37982		1.447701	-0.91727
	1.325701	-1.77307		1.414249	-0.86371		1.552454	-1.27142		0.764192	-0.31967		1.353059	-0.78099
	1.495812	-1.53547		1.276719	-0.69504		1.51257	-1.05589		0.646485	-0.25824		1.216186	-0.63586
	1.547891	-1.20333		1.073224	-0.51691		1.370622	-0.80317		0.508628	-0.193		1.020295	-0.47834
	1.332899	-0.75677		0.761866	-0.3184		1.050424	-0.49999		0.332728	-0.11823		0.723621	-0.29783

$n=15$						$n=20$					
k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
2.50	1.06102	-1.96779	**3.0**	1.030882	-1.98413	**0.25**	1.452584	-0.92579	**0.50**	1.548926	-1.34425
	1.140158	-1.92031		1.085415	-1.95389		1.361441	-0.79144		1.544948	-1.18215
	1.222981	-1.86211		1.152223	-1.91243		1.28445	-0.7031		1.516351	-1.06675
	1.303864	-1.79398		1.225577	-1.86011		1.215422	-0.63516		1.477783	-0.97357
	1.37882	-1.7162		1.300894	-1.79673		1.151453	-0.57894		1.433608	-0.89377
	1.444436	-1.62883		1.373929	-1.72185		1.090883	-0.53035		1.385612	-0.82301
	1.497445	-1.53186		1.440413	-1.63496		1.032638	-0.48712		1.33462	-0.75879
	1.534466	-1.42518		1.495812	-1.53547		0.975941	-0.4478		1.281013	-0.6995
	1.551754	-1.3086		1.535065	-1.42275		0.92019	-0.41145		1.224924	-0.64401
	1.544892	-1.18179		1.552245	-1.29604		0.864872	-0.37735		1.166326	-0.59152
	1.508304	-1.04423		1.540003	-1.15438		0.809516	-0.34499		1.10506	-0.54137
	1.434376	-0.89501		1.488557	-0.99648		0.753659	-0.31393		1.040848	-0.49303
	1.31156	-0.73241		1.383542	-0.82022		0.696804	-0.28379		0.973269	-0.44601
	1.119332	-0.55267		1.20035	-0.6214		0.638381	-0.25422		0.901726	-0.39986
	0.808757	-0.34456		0.882557	-0.38805		0.577688	-0.22487		0.825363	-0.35409
							0.51379	-0.19533		0.742922	-0.30813
							0.445346	-0.1651		0.652468	-0.26123
							0.370226	-0.13346		0.550762	-0.21226
							0.284545	-0.09918		0.431637	-0.15921
							0.179221	-0.05954		0.280087	-0.09744

Appendix – II (Continued)

Table 2.4 (Continued): Slope (β_i) and Intercept (α_i) in Modification to MLE by Method –II for n=20, 25

$n = 20$

k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
1.50	1.187756	-1.88808	**2.0**	1.091783	-1.95016	**2.5**	1.043742	-1.97727	**3.0**	1.02062	-1.9895
	1.296226	-1.80101		1.176436	-1.89603		1.101541	-1.94435		1.057474	-1.96976
	1.377265	-1.718		1.253626	-1.83782		1.163743	-1.90472		1.103538	-1.94315
	1.439361	-1.63654		1.322999	-1.77573		1.226912	-1.85908		1.155607	-1.91018
	1.486221	-1.55566		1.384175	-1.70989		1.288804	-1.8077		1.21132	-1.87092
	1.519844	-1.47489		1.436738	-1.64045		1.347648	-1.75071		1.26865	-1.82527
	1.541392	-1.39394		1.480221	-1.56749		1.401892	-1.68814		1.325701	-1.77307
	1.551541	-1.31262		1.514095	-1.4911		1.450079	-1.62001		1.380612	-1.7141
	1.550631	-1.23076		1.537748	-1.41133		1.49077	-1.5463		1.431484	-1.64813
	1.538742	-1.14823		1.550464	-1.32819		1.522475	-1.46697		1.476322	-1.5749
	1.51572	-1.0649		1.551384	-1.24167		1.5436	-1.38195		1.512967	-1.49413
	1.481176	-0.98059		1.539464	-1.15172		1.552367	-1.29112		1.539024	-1.40553
	1.434447	-0.89513		1.513404	-1.05824		1.546727	-1.19432		1.551747	-1.30873
	1.374521	-0.80824		1.471532	-0.96104		1.524217	-1.09133		1.547891	-1.20333
	1.299895	-0.71959		1.411631	-0.85982		1.481744	-0.98178		1.523452	-1.0888
	1.208311	-0.62862		1.330606	-0.75409		1.415206	-0.86514		1.47324	-0.96441
	1.096219	-0.53448		1.223859	-0.64301		1.318763	-0.7405		1.390056	-0.82907
	0.95758	-0.43559		1.083877	-0.52498		1.183233	-0.60617		1.02062	-1.9895
	0.780604	-0.32873		0.8964	-0.39656		0.991752	-0.45851		1.057474	-1.96976
	0.535268	-0.20511		0.625634	-0.24795		0.702999	-0.28701		1.103538	-1.94315

$n = 25$

k	α_i	β_i	k	α_i	β_i
0.25	1.472795	-0.96353	**0.50**	1.542745	-1.38675
	1.394275	-0.83491		1.551046	-1.23642
	1.327618	-0.75063		1.534571	-1.12968
	1.26795	-0.68604		1.508107	-1.04371
	1.212882	-0.63281		1.476171	-0.97029
	1.16104	-0.58702		1.440683	-0.90538
	1.11154	-0.54647		1.402595	-0.84666
	1.063764	-0.50983		1.362419	-0.79267
	1.017248	-0.47619		1.320436	-0.7424
	0.971626	-0.44491		1.276786	-0.69511
	0.926592	-0.41551		1.231522	-0.65025
	0.881875	-0.38764		1.18463	-0.6074
	0.837227	-0.36099		1.136041	-0.56619
	0.792405	-0.33532		1.085633	-0.52633
	0.74716	-0.31041		1.033232	-0.48754
	0.701225	-0.28608		0.9786	-0.44959
	0.654301	-0.26215		0.921421	-0.41223
	0.606035	-0.23841		0.861275	-0.3752
	0.555989	-0.21469		0.797593	-0.33824
	0.503598	-0.19074		0.72959	-0.301
	0.448085	-0.16629		0.656129	-0.26306
	0.388314	-0.14094		0.575481	-0.22383
	0.322467	-0.11412		0.484783	-0.18235
	0.247223	-0.08482		0.378638	-0.13693
	0.154858	-0.05077		0.244064	-0.08362

$n = 25$

k	α_i	β_i	k	α_i	β_i	k	α_i	β_i	k	α_i	β_i
1.50	1.162057	-1.90586	**2.0**	1.074676	-1.96008	**2.50**	1.033628	-1.98268	**3.0**	1.015	-1.9924
	1.258547	-1.83376		1.144742	-1.91734		1.078527	-1.95787		1.041982	-1.97821
	1.333187	-1.7656		1.210024	-1.87189		1.127562	-1.92835		1.076054	-1.95929
	1.3931	-1.69912		1.270342	-1.82383		1.178328	-1.89471		1.115115	-1.9361
	1.441428	-1.63342		1.325506	-1.77326		1.229329	-1.85721		1.157696	-1.90879
	1.479911	-1.56809		1.375313	-1.72026		1.27944	-1.81598		1.202595	-1.87737
	1.509618	-1.50286		1.419546	-1.66489		1.327723	-1.77107		1.248742	-1.84181
	1.53124	-1.43757		1.457966	-1.60721		1.373344	-1.72252		1.295128	-1.80201
	1.545227	-1.37209		1.49031	-1.54726		1.415521	-1.67034		1.340773	-1.75787
	1.551861	-1.30634		1.516286	-1.48509		1.453494	-1.61454		1.384693	-1.70928
	1.551296	-1.24024		1.535563	-1.42071		1.486504	-1.55509		1.425883	-1.65611
	1.543574	-1.17371		1.547765	-1.35413		1.51377	-1.49198		1.463294	-1.59822
	1.528641	-1.10668		1.552458	-1.28535		1.534466	-1.42518		1.495812	-1.53547
	1.506343	-1.03907		1.549138	-1.21436		1.547702	-1.35462		1.522232	-1.46772
	1.476422	-0.97079		1.537207	-1.14112		1.552494	-1.28026		1.541229	-1.39478
	1.438495	-0.90175		1.515946	-1.06556		1.547725	-1.20198		1.551312	-1.31648
	1.392028	-0.83179		1.484474	-0.98758		1.532098	-1.11968		1.55077	-1.23258
	1.336287	-0.76075		1.441689	-0.90706		1.504059	-1.03319		1.537585	-1.14283
	1.270258	-0.6884		1.386167	-0.82376		1.461685	-0.94225		1.509299	-1.0469
	1.192514	-0.61437		1.316001	-0.73738		1.402495	-0.84652		1.462807	-0.94434
	1.10097	-0.53817		1.228518	-0.6474		1.323121	-0.74546		1.393981	-0.8345
	0.992401	-0.45896		1.119706	-0.55297		1.218666	-0.63817		1.296951	-0.7164
	0.861366	-0.37526		0.982961	-0.45253		1.081278	-0.523		1.162521	-0.58828
	0.697315	-0.28405		0.805664	-0.3428		0.896274	-0.39648		0.973886	-0.44642
	0.473615	-0.17742		0.556094	-0.21474		0.627238	-0.24873		0.690088	-0.28032

Appendix-III

Table 2.5: Empirical Variances of $\hat{\lambda}$ based on MMLE: Method –I & Method –II

MMLE: Method –I						
n \ k	**0.25**	**0.50**	**1.50**	**2.0**	**2.5**	**3.0**
5	0.1341425	0.097691	0.044699	0.035495	0.02685	0.02321
10	0.099113	0.060441	0.020486	0.016022	0.012314	0.010444
15	0.07457	0.038852	0.013336	0.010474	0.00819	0.006875
20	0.059105	0.028732	0.009685	0.00823	0.005912	0.005017
25	0.0469	0.021249	0.007989	0.006167	0.004862	0.004187
MMLE: Method –II						
n \ k	**0.25**	**0.50**	**1.50**	**2.0**	**2.5**	**3.0**
5	0.130502	0.099553	0.040283	0.028477	0.019177	0.013697
10	0.097346	0.057394	0.018313	0.012291	0.008974	0.00647
15	0.079233	0.038871	0.011892	0.007887	0.005866	0.004383
20	0.064955	0.028415	0.008615	0.006064	0.00426	0.003162
25	0.053599	0.020655	0.006947	0.00485	0.003434	0.0025

Table 3.1: Correlation Coefficient between (z, h(z))

n	r	**k=3** Correlation Coefficient	n	r	**k=3** Correlation Coefficient	n	r	**k=3** Correlation Coefficient
5	2	0.998093	**20**	2	0.995589	**25**	2	0.995188
	3	0.998973		3	0.997319		3	0.997026
	4	0.999855		4	0.998184		4	0.997953
10	2	0.996872		5	0.998689		5	0.998499
	3	0.998236		6	0.999013		6	0.998853
	4	0.998887		7	0.999234		7	0.999096
	5	0.999246		8	0.999392		8	0.999272
	6	0.999457		9	0.999509		9	0.999403
	7	0.999574		10	0.999597		10	0.999504
	8	0.999587		11	0.999664		11	0.999583
	9	0.999936		12	0.999717		12	0.999646
15	2	0.996117		13	0.999757		13	0.999697
	3	0.997702		14	0.999786		14	0.999738
	4	0.998482		15	0.999806		15	0.999771
	5	0.99893		16	0.999814		16	0.999798
	6	0.999213		17	0.999802		17	0.99982
	7	0.999403		18	0.999734		18	0.999836
	8	0.999535		19	0.999958		19	0.999848
	9	0.999628					20	0.999853
	10	0.999693					21	0.99985
	11	0.999733					22	0.999829
	12	0.999745					23	0.999755
	13	0.999693					24	0.999961
	14	0.999951						

Appendix - IV

Table 3.2: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=0.25

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
0.25	5	2	3.473945	0.087458	**0.25**	25	2	7.652104	0.019929
		3	3.758664	0.093293			3	5.561148	0.03954
		4	4.879543	-0.2058			4	4.560551	0.058519
	10	2	4.171105	0.048719			5	4.000841	0.076474
		3	3.504577	0.090939			6	3.665968	0.092919
		4	3.354078	0.117855			7	3.463801	0.107262
		5	3.463016	0.114755			8	3.348766	0.118779
		6	3.769879	0.055961			9	3.296087	0.126585
		7	4.289639	-0.10754			10	3.291509	0.129589
		8	5.115904	-0.48441			11	3.326662	0.126444
		9	6.551342	-1.39176			12	3.396789	0.115459
	15	2	5.265932	0.032981			13	3.499593	0.094491
		3	4.07629	0.064297			14	3.634655	0.060776
		4	3.574889	0.092048			15	3.803218	0.010684
		5	3.365154	0.113558			16	4.008237	-0.06067
		6	3.318288	0.125085			17	4.254695	-0.16001
		7	3.383293	0.121334			18	4.550294	-0.29696
		8	3.540161	0.094621			19	4.906744	-0.48575
		9	3.785417	0.033383			20	5.342221	-0.7486
		10	4.128378	-0.08071			21	5.886358	-1.12252
		11	4.593546	-0.27881			22	6.591664	-1.67524
		12	5.231281	-0.6192			23	7.565199	-2.54986
		13	6.150807	-1.22836			24	9.095302	-4.1429
		14	7.655368	-2.466					
	20	2	6.443683	0.024857					
		3	4.793209	0.049064					
		4	4.029089	0.071939					
		5	3.627491	0.092594					
		6	3.414842	0.109882					
		7	3.317896	0.122332					
		8	3.301345	0.128043					
		9	3.347123	0.124542					
		10	3.44629	0.108555					
		11	3.595567	0.075658					
		12	3.795938	0.019719					
		13	4.052448	-0.06805					
		14	4.375019	-0.20077					
		15	4.780662	-0.39905					
		16	5.298423	-0.69753					
		17	5.981007	-1.16021					
		18	6.937358	-1.92385					
		19	8.463337	-3.37119					

Table 3.3: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=0.50

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
0.50	5	2	1.826788	-0.08609	**0.50**	25	2	1.159598	-0.00321
		3	2.415824	-0.31404			3	1.243444	-0.00985
		4	3.289925	-0.8439			4	1.330307	-0.02019
	10	2	1.400954	-0.02038			5	1.420469	-0.03454
		3	1.631698	-0.06555			6	1.514261	-0.05325
		4	1.890025	-0.14219			7	1.612065	-0.07677
		5	2.185062	-0.26078			8	1.714327	-0.10559
		6	2.531458	-0.43886			9	1.821572	-0.14035
		7	2.954923	-0.70845			10	1.934425	-0.18178
		8	3.507629	-1.13701			11	2.053629	-0.23079
		9	4.330148	-1.91053			12	2.180088	-0.28852
	15	2	1.266246	-0.00896			13	2.314904	-0.35637
		3	1.411523	-0.02804			14	2.45945	-0.43611
		4	1.566639	-0.05873			15	2.61546	-0.53007
		5	1.733364	-0.10301			16	2.785168	-0.64126
		6	1.914005	-0.16355			17	2.971528	-0.77378
		7	2.111638	-0.244			18	3.178554	-0.93333
		8	2.330478	-0.34953			19	3.411915	-1.12813
		9	2.576509	-0.48772			20	3.680009	-1.37068
		10	2.858645	-0.67022			21	3.996102	-1.68119
		11	3.191033	-0.91612			22	4.383204	-2.0956
		12	3.598311	-1.25975			23	4.887602	-2.68785
		13	4.130188	-1.7724			24	5.633568	-3.659
		14	4.920114	-2.64884					
	20	2	1.19952	-0.00502					
		3	1.305791	-0.01553					
		4	1.417061	-0.03206					
		5	1.533948	-0.05531					
		6	1.657199	-0.08611					
		7	1.787721	-0.12551					
		8	1.926631	-0.17482					
		9	2.075322	-0.23572					
		10	2.235557	-0.31042					
		11	2.409619	-0.40181					
		12	2.600527	-0.51382					
		13	2.812398	-0.65197					
		14	3.051053	-0.82425					
		15	3.325129	-1.04284					
		16	3.648297	-1.32758					
		17	4.0443	-1.71384					
		18	4.560933	-2.27466					
		19	5.326456	-3.20889					

Table 3.4: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=1.50

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
1.50	5	2	1.037595	-0.43145	**1.50**	25	2	0.418654	-0.09955
		3	1.390186	-0.74688			3	0.554401	-0.16364
		4	1.752863	-1.143			4	0.666795	-0.22453
	10	2	0.705847	-0.22912			5	0.765727	-0.28406
		3	0.929575	-0.37878			6	0.855712	-0.34306
		4	1.120058	-0.52797			7	0.939347	-0.40208
		5	1.29607	-0.68457			8	1.018326	-0.46155
		6	1.468208	-0.85582			9	1.093854	-0.52187
		7	1.646176	-1.05229			10	1.16685	-0.58341
		8	1.84409	-1.29439			11	1.238062	-0.64656
		9	2.096124	-1.63751			12	1.308135	-0.71176
	15	2	0.561846	-0.15872			13	1.377662	-0.77948
		3	0.740641	-0.26088			14	1.447222	-0.85033
		4	0.889665	-0.35945			15	1.51742	-0.925
		5	1.022675	-0.45807			16	1.588929	-1.00441
		6	1.146155	-0.55883			17	1.662549	-1.08973
		7	1.264116	-0.66359			18	1.739287	-1.1826
		8	1.37957	-0.77438			19	1.820503	-1.28532
		9	1.495225	-0.89375			20	1.90816	-1.40135
		10	1.614016	-1.02532			21	2.005341	-1.53631
		11	1.739774	-1.17472			22	2.117433	-1.70024
		12	1.1878524	-1.35181			23	2.255529	-1.9141
		13	2.042019	-1.57693			24	2.450671	-2.23697
		14	2.262397	-1.90695					
	20	2	0.476536	-0.12214					
		3	0.62963	-0.20063					
		4	0.756544	-0.27555					
		5	0.868683	-0.34931					
		6	0.971285	-0.42312					
		7	1.067408	-0.49783					
		8	1.1591	-0.57419					
		9	1.247893	-0.65298					
		10	1.335048	-0.73503					
		11	1.42171	-0.82134					
		12	1.509028	-0.91318					
		13	1.598284	-1.01219					
		14	1.691053	-1.12073					
		15	1.789478	-1.24225					
		16	1.8968	-1.38234					
		17	2.018568	-1.55092					
		18	2.166078	-1.76865					
		19	2.370786	-2.09366					

Table 3.5: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=2.0

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
2.0	5	2	0.91197	-0.48689	**2.0**	25	2	0.370266	-0.12903
		3	1.197436	-0.77718			3	0.495	-0.20352
		4	1.467311	-1.10273			4	0.596585	-0.2713
	10	2	0.628161	-0.27758			5	0.684557	-0.33524
		3	0.824474	-0.43362			6	0.763285	-0.39663
		4	0.984712	-0.57881			7	0.835287	-0.4563
		5	1.126877	-0.72222			8	0.902198	-0.51483
		6	1.260369	-0.87035			9	0.965167	-0.57269
		7	1.392817	-1.03116			10	1.025057	-0.63029
		8	1.534198	-1.21889			11	1.082551	-0.688
		9	1.707694	-1.47203			12	1.138216	-0.74622
	15	2	0.499965	-0.19869			13	1.192555	-0.80532
		3	0.661061	-0.31108			14	1.246035	-0.86576
		4	0.79155	-0.41369			15	1.29912	-0.92805
		5	0.904879	-0.51165			16	1.352304	-0.99283
		6	1.007294	-0.60758			17	1.406152	-1.06092
		7	1.102542	-0.7034			18	1.461356	-1.13342
		8	1.193281	-0.80092			19	1.518835	-1.21189
		9	1.281731	-0.90213			20	1.579904	-1.2987
		10	1.370112	-1.00963			21	1.646627	-1.39769
		11	1.46113	-1.12732			22	1.722641	-1.5158
		12	1.558884	-1.26195			23	1.81552	-1.66773
		13	1.671285	-1.42752			24	1.946813	-1.89572
		14	1.820208	-1.66392					
	20	2	0.422909	-0.15604					
		3	0.562606	-0.24524					
		4	0.675977	-0.32639					
		5	0.77411	-0.40316					
		6	0.862097	-0.47728					
		7	0.942886	-0.54987					
		8	1.01842	-0.62179					
		9	1.09011	-0.69382					
		10	1.159067	-0.76669					
		11	1.22625	-0.84119					
		12	1.292562	-0.91827					
		13	1.358955	-0.99909					
		14	1.426541	-1.08528					
		15	1.496786	-1.1792					
		16	1.571871	-1.28465					
		17	1.655519	-1.40842					
		18	1.755375	-1.56488					
		19	1.893085	-1.79525					

Table 3.6: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=2.5

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
2.5	5	2	0.823503	-0.51529	**2.5**	25	2	0.340376	-0.14995
		3	1.062195	-0.77978			3	0.455794	-0.23008
		4	1.272799	-1.05175			4	0.548491	-0.30086
	10	2	0.57515	-0.30804			5	0.627713	-0.36598
		3	0.750018	-0.46366			6	0.697703	-0.42714
		4	0.888111	-0.60149			7	0.760902	-0.48537
		5	1.006713	-0.73168			8	0.81889	-0.5414
		6	1.114436	-0.86044			9	0.872771	-0.59576
		7	1.217708	-0.9943			10	0.923362	-0.64891
		8	1.324215	-1.14403			11	0.971302	-0.70123
		9	1.451093	-1.33836			12	1.017109	-0.75307
	15	2	0.459334	-0.22543			13	1.061229	-0.8048
		3	0.605895	-0.34181			14	1.104064	-0.85677
		4	0.721962	-0.44405			15	1.145999	-0.90941
		5	0.82062	-0.53845			16	1.187427	-0.96319
		6	0.907894	-0.62807			17	1.228784	-1.01873
		7	0.987327	-0.71498			18	1.270591	-1.07684
		8	1.06135	-0.80089			19	1.313524	-1.13868
		9	1.131893	-0.8875			20	1.358547	-1.20596
		10	1.20077	-0.97686			21	1.407172	-1.28153
		11	1.27007	-1.07189			22	1.462075	-1.37057
		12	1.342836	-1.17758			23	1.528889	-1.4842
		13	1.42488	-1.3043			24	1.623858	-1.65491
		14	1.53241	-1.48217					
	20	2	0.388847	-0.17954					
		3	0.517299	-0.27401					
		4	0.619758	-0.35715					
		5	0.707012	-0.43362					
		6	0.784002	-0.50561					
		7	0.853577	-0.57445					
		8	0.917592	-0.64113					
		9	0.977372	-0.70644					
		10	1.033932	-0.77107					
		11	1.088117	-0.83572					
		12	1.140691	-0.90116					
		13	1.192417	-0.96829					
		14	1.244155	-1.03833					
		15	1.297001	-1.11301					
		16	1.352558	-1.19513					
		17	1.413555	-1.28972					
		18	1.485631	-1.40752					
		19	1.584981	-1.58001					

Table 3.7: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –I: k=3.0

k	n	r	δ_r	Υ_r
3.0	5	2	0.756006	-0.52925
		3	0.960339	-0.77042
		4	1.130073	-1.00107
	10	2	0.534895	-0.32762
		3	0.69274	-0.48003
		4	0.813998	-0.6099
		5	0.915303	-0.72823
		6	1.00469	-0.84113
		7	1.087804	-0.95427
		8	1.170898	-1.07623
		9	1.267362	-1.22952
	15	2	0.428985	-0.24373
		3	0.563638	-0.36103
		4	0.668287	-0.46109
		5	0.755655	-0.55109
		6	0.831562	-0.63447
		7	0.899384	-0.71341
		8	0.96139	-0.78958
		9	1.019314	-0.86453
		10	1.074712	-0.93995
		11	1.129285	-1.01818
		12	1.185431	-1.10308
		13	1.247665	-1.20273
		14	1.328662	-1.34098
	20	2	0.363827	-0.19613
		3	0.482922	-0.293
		4	0.576546	-0.37613
		5	0.655193	-0.45096
		6	0.723667	-0.52002
		7	0.78472	-0.58484
		8	0.840137	-0.64649
		9	0.891173	-0.70579
		10	0.938774	-0.76342
		11	0.983708	-0.82003
		12	1.026645	-0.87627
		13	1.068234	-0.9329
		14	1.109175	-0.99087
		15	1.150336	-1.05153
		16	1.192966	-1.11703
		17	1.239181	-1.19129
		18	1.293378	-1.28277
		19	1.368332	-1.41665
3.0	25	2	0.318752	-0.16504
		3	0.426364	-0.24819
		4	0.511761	-0.31999
		5	0.583937	-0.3848
		6	0.647017	-0.44465
		7	0.703372	-0.50073
		8	0.75453	-0.55387
		9	0.801555	-0.60468
		10	0.845229	-0.65364
		11	0.886153	-0.70113
		12	0.924814	-0.74752
		13	0.961616	-0.79313
		14	0.996919	-0.83829
		15	1.031056	-0.88335
		16	1.064358	-0.92869
		17	1.09718	-0.97481
		18	1.129937	-1.02234
		19	1.163158	-1.07216
		20	1.197591	-1.1256
		21	1.234406	-1.18488
		22	1.275687	-1.25405
		23	1.325857	-1.34195
		24	1.397816	-1.47476

Appendix - V

Table 3.8: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=0.25

k	n	r	δ_r	Υ_r
0.25	5	2	3.267468	0.132721
		3	3.44044	0.094053
		4	4.241724	-0.20182
	10	2	4.095068	0.08669
		3	3.414306	0.119652
		4	3.245847	0.135306
		5	3.329058	0.119951
		6	3.595149	0.05074
		7	4.04402	-0.11357
		8	4.73059	-0.457
		9	5.811663	-1.18651
	15	2	5.220663	0.061262
		3	4.025095	0.089076
		4	3.517373	0.11263
		5	3.300446	0.129289
		6	3.244853	0.13544
		7	3.298651	0.12607
		8	3.44044	0.094053
		9	3.664566	0.028894
		10	3.976672	-0.08561
		11	4.394443	-0.27606
		12	4.953784	-0.58979
		13	5.726488	-1.12145
		14	6.875098	-2.09775
	20	2	6.411821	0.047101
		3	4.758008	0.06952
		4	3.990497	0.090312
		5	3.585339	0.108583
		6	3.368808	0.1232
		7	3.267468	0.132721
		8	3.245758	0.13531
		9	3.285287	0.1286
		10	3.376677	0.109494
		11	3.516042	0.073859
		12	3.703498	0.016035
		13	3.94278	-0.07196
		14	4.241724	-0.20182
		15	4.613784	-0.39146
		16	5.081358	-0.67003
		17	5.683059	-1.08876
		18	6.491442	-1.7485
		19	7.666055	-2.88623
0.25	25	2	7.627645	0.038197
		3	5.534529	0.056735
		4	4.531775	0.074491
		5	3.96987	0.091067
		6	3.632716	0.105979
		7	3.428117	0.118636
		8	3.310425	0.128328
		9	3.254772	0.134187
		10	3.246787	0.135158
		11	3.277958	0.12994
		12	3.343353	0.116917
		13	3.44044	0.094053
		14	3.568497	0.058746
		15	3.728354	0.007603
		16	3.922384	-0.06389
		17	4.154731	-0.16187
		18	4.431818	-0.295
		19	4.763314	-0.47585
		20	5.163908	-0.72359
		21	5.656724	-1.06909
		22	6.2805	-1.56615
		23	7.106836	-2.3194
		24	8.291435	-3.57152

Table 3.9: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=0.50

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
0.50	5	2	1.775421	-0.13818	0.50	25	2	1.158527	-0.00616
		3	2.292353	-0.36648			3	1.241604	-0.01418
		4	2.988413	-0.82167			4	1.327531	-0.02581
	10	2	1.391967	-0.03671			5	1.41658	-0.04135
		3	1.614299	-0.0881			6	1.509064	-0.06115
		4	1.860575	-0.16884			7	1.605344	-0.08563
		5	2.138313	-0.28809			8	1.705841	-0.11527
		6	2.459047	-0.46074			9	1.811046	-0.15066
		7	2.841793	-0.71263			10	1.921537	-0.19251
		8	3.321357	-1.09387			11	2.038004	-0.24168
		9	3.973092	-1.72028			12	2.161274	-0.29924
	15	2	1.262809	-0.01674			13	2.292353	-0.36648
		3	1.405268	-0.03918			14	2.432481	-0.44507
		4	1.556705	-0.07278			15	2.583211	-0.53714
		5	1.718723	-0.11937			16	2.746521	-0.64544
		6	1.893383	-0.1814			17	2.92499	-0.77367
		7	2.083383	-0.26222			18	3.122069	-0.9269
		8	2.292353	-0.36648			19	3.342526	-1.1123
		9	2.52533	-0.50088			20	3.593223	-1.34053
		10	2.7896	-0.67546			21	3.884577	-1.62822
		11	3.09627	-0.90616			22	4.233591	-2.00349
		12	3.463551	-1.22021			23	4.671077	-2.51926
		13	3.924596	-1.66958			24	5.263125	-3.29443
		14	4.550783	-2.37251					
	20	2	1.197748	-0.00955					
		3	1.302675	-0.0221					
		4	1.412264	-0.04052					
		5	1.527098	-0.06544					
		6	1.647871	-0.09767					
		7	1.775421	-0.13818					
		8	1.910768	-0.18823					
		9	2.055176	-0.24937					
		10	2.210227	-0.32363					
		11	2.377941	-0.41367					
		12	2.56096	-0.52307					
		13	2.762818	-0.65673					
		14	2.988413	-0.82167					
		15	3.244802	-1.02832					
		16	3.542721	-1.2931					
		17	3.899716	-1.64382					
		18	4.347643	-2.13316					
		19	4.954907	-2.88001					

Table 3.10: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=1.50

k	n	r	δ_r	Υ_r
1.50	5	2	1.075006	-0.51634
		3	1.374915	-0.78692
		4	1.675123	-1.1163
	10	2	0.757851	-0.2895
		3	0.956607	-0.4247
		4	1.131581	-0.56311
		5	1.294999	-0.7093
		6	1.454811	-0.86867
		7	1.618304	-1.04925
		8	1.795003	-1.2653
		9	2.002707	-1.54842
	15	2	0.610889	-0.20436
		3	0.771343	-0.29794
		4	0.909809	-0.39078
		5	1.035201	-0.4846
		6	1.152455	-0.58086
		7	1.264848	-0.68107
		8	1.374915	-0.78692
		9	1.484949	-0.9006
		10	1.597383	-1.02512
		11	1.715246	-1.16503
		12	1.842967	-1.32783
		13	1.988195	-1.52758
		14	2.167279	-1.79612
	20	2	0.521653	-0.15885
		3	0.659807	-0.2313
		4	0.778339	-0.30237
		5	0.884638	-0.37307
		6	0.982692	-0.44417
		7	1.075006	-0.51634
		8	1.163325	-0.59021
		9	1.248989	-0.66645
		10	1.333112	-0.74581
		11	1.416712	-0.82917
		12	1.500803	-0.91764
		13	1.586495	-1.01269
		14	1.675123	-1.1163
		15	1.768437	-1.23139
		16	1.868956	-1.36241
		17	1.9807	-1.51689
		18	2.110952	-1.70897
		19	2.27564	-1.97083
1.50	25	2	0.460274	-0.13029
		3	0.583236	-0.18976
		4	0.688548	-0.24778
		5	0.782621	-0.30509
		6	0.868898	-0.36218
		7	0.949507	-0.41947
		8	1.025894	-0.47731
		9	1.099116	-0.53604
		10	1.169994	-0.59599
		11	1.239203	-0.65751
		12	1.30733	-0.72101
		13	1.374915	-0.78692
		14	1.442485	-0.85578
		15	1.510586	-0.92824
		16	1.579818	-1.0051
		17	1.650884	-1.08742
		18	1.72465	-1.17661
		19	1.802254	-1.27464
		20	1.885286	-1.3844
		21	1.976123	-1.51037
		22	2.078649	-1.66011
		23	2.199996	-1.84789
		24	2.355862	-2.10621

Table 3.11: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=2.0

k	n	r	δ_r	Υ_r
2.0	5	2	0.950885	-0.56721
		3	1.191522	-0.81173
		4	1.416142	-1.08187
	10	2	0.679382	-0.34035
		3	0.851468	-0.47838
		4	0.997524	-0.61126
		5	1.129003	-0.74399
		6	1.252859	-0.88122
		7	1.374824	-1.02886
		8	1.501658	-1.1967
		9	1.645321	-1.40606
	15	2	0.548399	-0.24827
		3	0.691257	-0.34925
		4	0.811518	-0.44458
		5	0.917732	-0.53683
		6	1.014589	-0.62777
		7	1.105111	-0.71891
		8	1.191522	-0.81173
		9	1.275696	-0.90789
		10	1.359475	-1.00956
		11	1.445006	-1.11985
		12	1.535293	-1.24382
		13	1.635435	-1.39096
		14	1.75637	-1.58302
	20	2	0.467561	-0.19701
		3	0.592316	-0.2779
		4	0.697404	-0.35389
		5	0.789881	-0.42673
		6	0.873578	-0.49755
		7	0.950885	-0.56721
		8	1.023444	-0.63642
		9	1.092478	-0.70582
		10	1.158965	-0.77607
		11	1.223755	-0.84786
		12	1.287644	-0.92202
		13	1.351459	-0.99957
		14	1.416142	-1.08187
		15	1.482891	-1.17089
		16	1.553392	-1.26964
		17	1.630321	-1.3832
		18	1.718545	-1.52121
		19	1.828849	-1.70597
2.0	25	2	0.411493	-0.164
		3	0.523443	-0.23198
		4	0.617976	-0.29579
		5	0.701163	-0.35674
		6	0.776299	-0.41568
		7	0.84543	-0.4732
		8	0.90994	-0.5298
		9	0.97083	-0.58586
		10	1.028866	-0.64174
		11	1.084659	-0.69778
		12	1.138726	-0.75433
		13	1.191522	-0.81173
		14	1.243472	-0.87039
		15	1.294995	-0.93077
		16	1.346532	-0.99345
		17	1.398581	-1.05915
		18	1.451735	-1.12881
		19	1.506763	-1.20378
		20	1.564722	-1.28598
		21	1.627195	-1.37846
		22	1.696778	-1.48637
		23	1.778282	-1.61956
		24	1.882446	-1.80076

Table 3.12: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=2.5

k	n	r	δ_r	Υ_r
2.5	5	2	0.861412	-0.59062
		3	1.061074	-0.81049
		4	1.236938	-1.03581
	10	2	0.624105	-0.37079
		3	0.775859	-0.5065
		4	0.90096	-0.63146
		5	1.010278	-0.75121
		6	1.110135	-0.87009
		7	1.205369	-0.99288
		8	1.301218	-1.1269
		9	1.406477	-1.28769
	15	2	0.505884	-0.2765
		3	0.634713	-0.37973
		4	0.741024	-0.47385
		5	0.833067	-0.56212
		6	0.915332	-0.64663
		7	0.99066	-0.72896
		8	1.061074	-0.81049
		9	1.128203	-0.89265
		10	1.193554	-0.97711
		11	1.25879	-1.0662
		12	1.326139	-1.16362
		13	1.399321	-1.27627
		14	1.486334	-1.42021
	20	2	0.431927	-0.22253
		3	0.545758	-0.3072
		4	0.640197	-0.38439
		5	0.722055	-0.45645
		6	0.795028	-0.52485
		7	0.861412	-0.59062
		8	0.922769	-0.65454
		9	0.980241	-0.71727
		10	1.03472	-0.77944
		11	1.086952	-0.84163
		12	1.137612	-0.90453
		13	1.187364	-0.96891
		14	1.236938	-1.03581
		15	1.287228	-1.10664
		16	1.339472	-1.1836
		17	1.395615	-1.27039
		18	1.459215	-1.37414
		19	1.53824	-1.51153

k	n	r	δ_r	Υ_r
2.5	25	2	0.380247	-0.18713
		3	0.483127	-0.25948
		4	0.568948	-0.32557
		5	0.643548	-0.38724
		6	0.710111	-0.44563
		7	0.77061	-0.50152
		8	0.826378	-0.55548
		9	0.878372	-0.60798
		10	0.927315	-0.6594
		11	0.973778	-0.71008
		12	1.018229	-0.76034
		13	1.061074	-0.81049
		14	1.102677	-0.86088
		15	1.143386	-0.91187
		16	1.183553	-0.96389
		17	1.223562	-1.01748
		18	1.26386	-1.07334
		19	1.305013	-1.13244
		20	1.347792	-1.19618
		21	1.393349	-1.2668
		22	1.443579	-1.34808
		23	1.502026	-1.44738
		24	1.576698	-1.58188

Table 3.13: Slope (δ_r) and Intercept (Υ_r) in Modification to MLE from Censored Sample by Method –II: k=3.0

k	n	r	δ_r	Υ_r	k	n	r	δ_r	Υ_r
3.0	5	2	0.792232	-0.59996	**3.0**	25	2	0.357065	-0.20337
		3	0.961963	-0.7983			3	0.452459	-0.27784
		4	1.103985	-0.98907			4	0.531195	-0.34448
	10	2	0.581344	-0.38937			5	0.59893	-0.40557
		3	0.717204	-0.52084			6	0.658748	-0.46247
		4	0.826482	-0.63772			7	0.712562	-0.5161
		5	0.91957	-0.74603			8	0.761658	-0.56713
		6	1.002343	-0.84997			9	0.806955	-0.61606
		7	1.079056	-0.95366			10	0.849143	-0.66331
		8	1.153999	-1.06288			11	0.888761	-0.70923
		9	1.234031	-1.18959			12	0.926244	-0.75413
	15	2	0.473417	-0.29512			13	0.961963	-0.7983
		3	0.590945	-0.39819			14	0.996242	-0.84204
		4	0.686315	-0.48965			15	1.029382	-0.88566
		5	0.767512	-0.57335			16	1.061682	-0.9295
		6	0.838856	-0.65164			17	1.093453	-0.974
		7	0.903046	-0.72617			18	1.125052	-1.01969
		8	0.961963	-0.7983			19	1.156921	-1.06732
		9	1.017069	-0.86929			20	1.189658	-1.11796
		10	1.069661	-0.94055			21	1.224149	-1.17333
		11	1.121098	-1.01389			22	1.261861	-1.23637
		12	1.173135	-1.09218			23	1.035553	-1.31285
		13	1.228647	-1.18071			24	1.361532	-1.41644
		14	1.293826	-1.29196					
	20	2	0.405112	-0.24001					
		3	0.510002	-0.32609					
		4	0.595903	-0.40277					
		5	0.669416	-0.47292					
		6	0.734121	-0.53826					
		7	0.792232	-0.59996					
		8	0.845244	-0.65887					
		9	0.894237	-0.7157					
		10	0.940041	-0.77103					
		11	0.983333	-0.82541					
		12	1.024706	-0.87942					
		13	1.064725	-0.9337					
		14	1.103985	-0.98907					
		15	1.143196	-1.04661					
		16	1.18332	-1.10802					
		17	1.225857	-1.17613					
		18	1.273558	-1.25648					
		19	1.332648	-1.3622					

Appendix- VI

Table 3.14: Empirical Variances of MMLE of λ from Censored Sample: Method – I & Method –II

MMLE: Method –I						
k / (n,r)	**0.25**	**0.50**	**1.50**	**2.0**	**2.5**	**3.0**
(5,4)	11.49365	0.412213	0.18131	0.187485	0.21507	0.245198
(10,8)	0.410329	0.11826	0.103083	0.125636	0.150761	0.182708
(15,12)	0.138162	0.075133	0.040271	0.109317	0.131422	0.16653
(20,16)	0.092643	0.055522	0.071039	0.091347	0.12222	0.150345
(25,20)	0.067846	0.042435	0.067635	0.089057	0.114548	0.150973
MMLE: Method –II						
k / (n,r)	**0.25**	**0.50**	**1.50**	**2.0**	**2.5**	**3.0**
(5,4)	5.027481	0.388384	0.179044	0.193406	0.21721	0.737996
(10,8)	0.534766	0.103742	0.100408	0.120603	0.151993	0.180261
(15,12)	0.180459	0.065647	0.082163	0.102875	0.130453	0.162365
(20,16)	0.109447	0.047965	0.072703	0.092277	0.122778	0.149991
(25,20)	0.083155	0.038627	0.066796	0.084847	0.113837	0.146095

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