A short note on the paper (PJSOR) "New Characterizations of the Pareto Distribution"

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Abstract

Nofal and El Gebaly (2017), presented certain characterizations of the Pareto distribution based on the conditional expectations of power of the order statistics. In this very short note we show that the same results can easily be obtained in terms of the power of the random variable.

Keywords: Truncated moments ; Pareto distribution ; characterizations.

1. Introduction

The pdf's (probability density function) and cdf's (cumulative distribution function) of the Pareto distributions of the first and second types (or kinds), using the notation of Nofal and El Gebaly (2017), are given, respectively, by

$$f_1(x) = \beta \alpha^{\beta} x^{-(\beta+1)}, \qquad x > \alpha,$$
$$F_1(x) = 1 - \alpha^{\beta} x^{-\beta}, \quad x \geq \alpha, \tag{1}$$

and

$$f_2(x) = k\theta^{k} (x+\theta)^{-(k+1)}, \qquad x > 0,$$
$$F_2(x) = 1 - \theta^{k} (x+\theta)^{-k}, \quad x \geq 0, \tag{2}$$

where α, β, θ and k are positive parameters.

Let $X_1, X_2, \dots, X_n$ be a sample of size n from (1) (or (2)) and let $X_{i:n}$, $i = 1,2,\dots,n$ be the corresponding order statistics. Nofal and El Gebaly (2017) stated the following four Theorems:

Theorem 1 (NE). Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}$, density function $f(\cdot)$ and Failure (hazard) rate function $h(\cdot)$. Let $X_{i:n}$, $i = 1,2,\dots,n$ denote the order statistics of a random sample of size n from $F(\cdot)$. The random variable X has the Pareto distribution of the first type if and only if

$$E[X^{s}_{r+1:n} | X_{r:n} = x] =$$

$$x^s + \frac{sx^{s+1}h(x)}{\beta^2(n-r)-s\beta}, s = 1,2,\ldots, r = 1,2,\ldots,n-1, h(x) = \frac{\beta}{x}. \quad (3)$$

Theorem 2 (NE). Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}$, density function $f(\cdot)$ and Failure (hazard) rate function $h(\cdot)$. Let $X_{i:n}$, $i = 1,2,\ldots,n$ denote the order statistics of a random sample of size n from $F(\cdot)$. The random variable X has the Pareto distribution of the second type if and only if

$$E[X^s_{r+1:n}|X_{r:n} = x] =$$

$$\sum_{j=0}^{s} \frac{m_{(1)}s!x^{s-j}\theta^j\left(1+\frac{x}{\theta}\right)^j}{(s-j)!m_{(j+1)}}, s = 1,2,\ldots, r = 1,2,\ldots,n-1, \quad (4)$$

where $m_{(r)} = m(m-1)(m-2)\ldots(m-r+1)$ and $m = k(n-r)$.

Remarks 1. (A) Clearly, equation (3) can be written in a simple form of

$$E[X^s_{r+1:n}|X_{r:n}x]\left(\frac{\beta(n-r)}{\beta(n-r)-s}\right)x^s. \quad (5)$$

The above formula can also be obtained by combining equations (2.2) and (2.3) of Nofal and El Gebaly. The condition " $s < \beta(n-r)$ ", however, is missing in the statement of Theorem 1.

(B) Equation (4) has the following simpler form

$$E[X^s_{r+1:n}|X_{r:n} = x] =$$

$$m\left(\sum_{j=0}^{s} \frac{s_{(j)}\left(1+\frac{\theta}{x}\right)^j}{m_{(j+1)}}\right)x^s, s = 1,2,\ldots, r = 1,2,\ldots,n-1. \quad (6)$$

The condition " $s < k(n-r)+1$ " is missing in the statement of Theorem 2.

(C) For these types of characterizations there in no need to employ the order statistics.

Theorem 3 (NE). Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}$, density function $f(\cdot)$ and Failure (hazard) rate function $h(\cdot)$, $E(X^j) = \mu^j$. The random variable X has the Pareto distribution of the first type if and only if

$$E\left[X^j|X \geq x\right] = \mu^j \frac{x^{j+1}}{\beta\alpha^j}h(x), \quad j < \beta. \quad (7)$$

Theorem 4 (NE). Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}$, density function $f(\cdot)$ and reversed Failure (hazard) rate function $\eta(\cdot)$. Then X has the Pareto distribution of the first type if and only if

$$E\left[X^j|X \leq x\right] = \mu^j + \frac{x}{\beta-j}\left(\alpha^j - x^j\right)\eta(x). \quad (8)$$

Remarks 2. (D) The correct form of (7) is

$$E\left[X^j|X \geq x\right] = \frac{x^{j+1}}{\beta-j}h(x), \quad j < \beta. \tag{9}$$

The condition " $f(\cdot)$ is differentiable " is missing in Theorem 3.

(E) Clearly, equation (9) can be written in a simple form of an integral equation

$$\int_x^\infty u^j f(u)du = \frac{x^{j+1}}{\beta-j}f(x). \tag{10}$$

(F) The correct form of equation (8) is

$$E\left[X^j|X \leq x\right] = \frac{\beta\alpha^j}{\beta-j} + \frac{x}{\beta-j}\left(\alpha^j - x^j\right)\eta(x). \tag{11}$$

The condition " $f(\cdot)$ is differentiable " is missing in Theorem 4.

(G) Clearly, equation (11) can be written in a simple form of an integral equation

$$\int_\alpha^x \left((\beta-j)u^j - \beta\alpha^j\right)f(x)du = x\left(\alpha^j - x^j\right)f(x). \tag{12}$$

2. Characterizations of the Pareto Distributions

We can combine Theorems 1 and 3 and present the following Proposition, which is much simpler, to characterize Pareto distribution of the first type in terms of the truncated moment of the power of the random variable.

Proposition 1. Let $X: \Omega \to (\alpha, \infty)$, $\alpha > 0$, be a continuous random variable with cdf $F(\cdot)$ and pdf $f(\cdot)$. The random variable X has pdf (cdf) (1) if and only if

$$E\left[X^s|X \geq x\right] = \left(\frac{\beta}{\beta-s}\right)x^s, \quad s < \beta. \tag{13}$$

Proof. If X has pdf given in (1), then

$$E\left[X^s|X \geq x\right] = \left(1 - F_1(x)\right)\int_x^\infty u^s f_1(u)du =$$

$$\alpha^{-\beta}x^\beta\beta\alpha^\beta\int_x^\infty u^{s-\beta-1}du = \left(\frac{\beta}{\beta-s}\right)x^s, for \ s < \beta.$$

Conversely, if (13) holds, then

$$\left(1 - F_1(x)\right)^{-1}\int_x^\infty u^s f_1(u)du = \left(\frac{\beta}{\beta-s}\right)x^s, \quad s < \beta,$$

or

$$\int_x^\infty u^s f_1(u)du = \left(1 - F_1(x)\right)\left(\frac{\beta}{\beta-s}\right)x^s.$$

Taking derivatives from both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f_1(x)}{1 - F_1(x)} = \frac{\beta}{x}.$$

Now, integrating both sides of the last equation from α to x, we have the cdf given in (1).

Remark 3. (H) Clearly, the function X^s in Proposition 1 can be replaced with a function of X, $\psi(X)$.

The following Proposition is a simple alternative to Theorem 2.

Proposition 2. Let $X:\Omega \rightarrow (0,\infty)$, be a continuous random variable with cdf $F(\cdot)$ and pdf $f(\cdot)$. The random variable X has pdf (cdf) (2) if and only if

$$E[(X+\theta)^s | X \geq x] = \frac{1}{k-s}(x+\theta)^{s-k}, \quad s<k. \tag{14}$$

Proof. If X has pdf given in (2), then

$$E[(X+\theta)^s | X \geq x] = \theta^{-k}(x+\theta)^k \int_x^\infty k\theta^k (u+\theta)^s (u+\theta)^{-(k+1)} du$$
$$= \frac{1}{k-s}(x+\theta)^{s-k}, \qquad for \;\; s<\beta.$$

Conversely, if (14) holds, then

$$\int_x^\infty (u+\theta)^s f_2(u) du = \frac{1}{k-s}(x+\theta)^{s-k}\big(1-F_2(x)\big).$$

Taking derivatives from both sides of the above equation with respect to x and rearranging the terms, we arrive at

$$\frac{f_2(x)}{1-F_2(x)} = \frac{k}{x+\theta}.$$

Now, integrating both sides of the last equation from 0 to x, we have the cdf given in (2).

Remark 4. (I) Clearly, the function $(X+\theta)^s$ in Proposition 2 can be replaced with a function of X , $\psi(X)$.

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Reference

1. Nofal, Z.M and El Gebaly, Y.M. (2017). New characterizations of the Pareto distribution, pak.j.stat.oper.res., Vol. XIII No.1 2017 pp 63-74.