Improved Ratio-type Estimators of Population Mean in Ranked Set Sampling Using Two Concomitant Variables

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Abstract

In this paper, we propose an efficient class of ratio-in-exponential-type estimators with two concomitant variables using Ranked Set Sampling (*RSS*) scheme which improves the available estimators. The biases and mean square errors (*MSEs*) of the proposed estimators are obtained up to first degree approximation. Comparisons among the proposed and competitor estimators are made both theoretically and through simulation study. It turned out that when the variable of interest and the concomitant variables jointly followed a trivariate Gamma distribution, the proposed class of estimators dominates all other competitor estimators.

Keywords: Ranked set sampling, *MSE,* Bias, Concomitant variables.

Mathematics subject classification 62D05.

1. Introduction

Ranked set sampling (*RSS*) is a sampling technique which is used to reduce cost and increase efficiency in that situation where the measurement of survey variable is costly and time consuming, but it can be ranked easily at no cost or at very little cost. The technique of *RSS* was first introduced by Mclntyre (1952) to increase efficiency of the estimator of the population mean. The general method of a *RSS* can be described as follows: First, *m* subsamples, each of size *m*, are drawn at random from a population. Next, for each sub sample, the elements in the subsample are ranked relating to the concomitant variables and then one and only one element of the subsample ranked is measured. The procedure produces a sample of n measurement of independent order statistics.

Takahasi and Wakimoto (1968) proved the mathematical theory that the sample mean under *RSS* is an unbiased estimator of the finite population mean and more precise than the sample mean estimator under simple random sampling (*SRS*). In some situations, ranking may not be done perfectly. To tackle this problem, Stokes (1977) considered the case where the ranking of elements is done on basis of the auxiliary variable *X* instead of judgment. Singh et al. (2014) proposed an estimator for population mean and ranking of the elements is observed on the basis of auxiliary variable. The use of the auxiliary information plays an important role in increasing efficiency of the estimators. Samawi and muttlak (1996) have suggested an estimator for population ratio in *RSS* and showed that it has less variance as compared to usual ratio estimator in *SRS*. Khan and Shabbir (2015) suggested a class of Hartley-Ross type unbiased estimator in *RSS*. Khan and Shabbir (2016) have also suggested Hartley-Ross type unbiased estimators in *RSS* and stratified ranked set sampling (*SRSS*). Khan et al. (2016) proposed unbiased ratio estimator of finite population mean in *SRSS*.

Munoz and Rueda (2009) used relative bias (*RB*) and the relative root mean square error (*RRMSE*), for the comparison of different estimators. For the more detail see Chambers and Dunstan (1986), Rao et al. (1990), Silva and Skinner (1995) and Harms and Duchesne (2006).

In this article, we investigate the properties of the usual mean estimator in *RSS* and propose an efficient class of the ratio-in-exponential type estimators using two concomitant variables under *RSS* scheme.

2. Ranked set sampling procedure with two concomitant variables

In ranked set sampling *m* independent random samples each of size *m* are chosen and the items in each sample are selected with equal probability and without replacement from a finite population of size *N*. The items of each random sample are ranked with respect to the characteristic of the study variable or concomitant variable. Let Y be the study variable and *X* and *Z* are the two concomitant variables. Then randomly select m^2 trivariate sample elements from the population and allocate them into m sets, each of size m. Each sample is ranked with respect to one of the concomitant variable *X* or *Z*. Here, ranking is done on the basis of the concomitant variable *X*. An actual measurement from the first sample is then taken on the item with the smallest rank of *X*, together with variable *Y* and *Z* associated with smallest rank of *X*. From the second sample of size m, the variables *Y* and *Z* associated with the second smallest rank of X are measured. By this way, this procedure is continued until, the *Y* and *Z* values associated with the highest rank of *X* are measured from the m^{th} sample. This completes one cycle of sampling process. The procedure is repeated r times to obtain a sample of size $n = mr$ items. Thus in a *RSS* scheme, a total of $m^2 r$ items have been drawn from the population and only mr of them are selected for analysis. To estimate the population mean, $\left(\bar{Y}\right)$, in RSS using ratio-type estimator with two concomitant variables, the procedure of selecting *n* ranked set samples can be summarized as follows:

Step 1: Randomly select m^2 trivariate sample items from the population.

Step 2: Allocate these m^2 items into m sets, each of size m .

Step3: Each set is ranked with respect to the concomitant variable *X*.

Step 4: Select the i^{th} ranked item from the i^{th} $\left(i = 1, 2, ..., m\right)$ set for actual magnitude.

Step 5: Repeat steps 1 through 4 for r cycles until the desired sample size $n = mr$, is obtained.

The usual RSS mean estimator $\bar{y}_{(RSS)}$ and its variance, are given by

$$\bar{y}_{(RSS)} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} y_{[i:m]j}, \tag{1}$$

$$V(\bar{y}_{(RSS)}) = \bar{Y}^2\left(\gamma C_y^2 - W_{[y]}^2\right), \tag{2}$$

where $W_{[y]}^2 = \frac{1}{m^2 r\bar{Y}^2}\sum_{i=1}^{m}\mathrm{T}_{y[i:m]}^2$, $\mathrm{T}_{y[i:m]} = \left(\mu_{y[i:m]} - \bar{Y}\right)$, $\gamma = \left(\frac{1}{mr}\right)$ and C_y is the coefficient of variation of *y*. The value of $\mu_{y[i:m]}$ depends on order statistics from some specific distribution (see Arnold et al, 1992).

3. Proposed class of estimators in RSS

We propose a class of ratio-in-exponential type estimators having two concomitant variables X and Z in *RSS* as

$$\bar{y}_{G(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{X}}{\bar{x}_{(rss)}}\right)^{\alpha_1}\left(\frac{\bar{Z}}{\bar{z}_{[rss]}}\right)^{\alpha_2}\left[k\exp\left(\frac{\bar{X}-\bar{x}_{(rss)}}{\bar{X}+\bar{x}_{(rss)}}\right) + (1-k)\exp\left(\frac{\bar{Z}-\bar{z}_{[rss]}}{\bar{Z}+\bar{z}_{[rss]}}\right)\right] \tag{3}$$

where

$$\bar{y}_{[rss]} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} y_{[i:m]j}, \quad \bar{x}_{(rss)} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} x_{(i:m)j}, \quad \bar{z}_{[rss]} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} z_{[i:m]j},$$

α_1 and α_2 are unknown constant whose values are to be determined so that *MSE* of $\bar{y}_{G(RSS)}$ is minimized and *k* is a scalar quantity which can take 0 or 1 values. Also $\bar{X}$ and $\bar{Z}$ are the population means of X and Z respectively.

To find the bias and *MSE* of the estimators, we define the following error terms:

Let $\bar{y}_{[rss]} = \bar{Y}\left(1+e_0\right)$, $\bar{x}_{(rss)} = \bar{X}\left(1+e_1\right)$, $\bar{z}_{[rss]} = \bar{Z}\left(1+e_2\right)$

Such that $E\left(e_p\right) = 0$, (p=0, 1, 2,), and

$$E\left({e_0}^2\right) = \gamma C_y^2 - W_{[y]}^2 = V_{200},\ E\left(e_1^2\right) = \gamma C_x^2 - W_{[x]}^2 = V_{020},\quad E\left(e_2^2\right) = \gamma C_z^2 - W_{[z]}^2 = V_{002},$$

$$E\left(e_0 e_1\right) = \gamma C_{yx} - W_{(yx)} = V_{100},\ E\left(e_0 e_2\right) = \gamma C_{yz} - W_{[yz]} = V_{101},\ E\left(e_1 e_2\right) = \gamma C_{xz} - W_{(xz)} = V_{011},$$

$$W_{[y]}^2 = \frac{1}{m^2 r\bar{Y}^2}\sum_{i=1}^{m}\mathrm{T}_{y[i:m]}^2,\quad W_{(x)}^2 = \frac{1}{m^2 r\bar{X}^2}\sum_{i=1}^{m}\mathrm{T}_{x(i:m)}^2,\quad W_{[z]}^2 = \frac{1}{m^2 r\bar{Z}^2}\sum_{i=1}^{m}\mathrm{T}_{z[i:m]}^2$$

$$W_{(yx)} = \frac{1}{m^2 r\bar{Y}\bar{X}}\sum_{i=1}^{m}\mathrm{T}_{yx(i:m)}^2,\quad W_{[yz]} = \frac{1}{m^2 r\bar{Y}\bar{Z}}\sum_{i=1}^{m}\mathrm{T}_{yz[i:m]}^2,\quad W_{(xz)} = \frac{1}{m^2 r\bar{X}\bar{Z}}\sum_{i=1}^{m}\mathrm{T}_{xz(i:m)}^2$$

$$\mathrm{T}_{y[i:m]} = \left(\mu_{y[i:m]} - \bar{Y}\right),\quad \mathrm{T}_{x(i:m)} = \left(\mu_{x(i:m)} - \bar{Y}\right),\quad \mathrm{T}_{z[i:m]} = \left(\mu_{z[i:m]} - \bar{Y}\right)$$

$$\mathrm{T}_{yx(i:m)} = \left(\mu_{y[i:m]} - \bar{Y}\right)\left(\mu_{x(i:m)} - \bar{X}\right),\quad \mathrm{T}_{yz[i:m]} = \left(\mu_{y[i:m]} - \bar{Y}\right)\left(\mu_{z[i:m]} - \bar{Z}\right),$$

$$\mathrm{T}_{xz(i:m)} = \left(\mu_{x(i:m)} - \bar{X}\right)\left(\mu_{z[i:m]} - \bar{Z}\right).$$

Here $C_{yx}=\rho_{yx}C_xC_y$, $C_{yz}=\rho_{yz}C_yC_z$, $C_{xz}=\rho_{xz}C_xC_z$, where C_x, C_y and C_z are the coefficients of variation of X, Y and Z respectively. The values of $\mu_{x[i:m]}$ and $\mu_{z[i:m]}$ depend on order statistics from some specific distributions.

In terms of e's up to first order of approximation, we have

$$\bar{y}_{G(RSS)}=\bar{Y}\left[1+e_0-\alpha_1e_1-\alpha_1e_0e_1+\frac{1}{2}\alpha_1\left(\alpha_1+1\right)e_1^2\right]\left[1-\alpha_2e_2+\frac{1}{2}\alpha_2\left(\alpha_2+1\right)e_2^2\right]$$
$$\left[k\left(1-\frac{1}{2}e_1+\frac{3}{8}e_1^2\right)+\left(1-k\right)\left(1-\frac{1}{2}e_2+\frac{3}{8}e_2^2\right)\right],$$

or

$$(\bar{y}_{G(RSS)}-\bar{Y})=\bar{Y}\begin{bmatrix} e_0-\frac{1}{2}\left(k+2\alpha_1\right)e_1-\frac{1}{2}\left(1-k+2\alpha_2\right)e_2+\frac{1}{8}\left\{k\left(4\alpha_1+3\right)+4\alpha_1\left(\alpha_1+1\right)\right\}e_1^2 \\ +\frac{1}{8}\left\{(1-k)\left(4\alpha_2+1\right)+4\alpha_2\left(\alpha_2+1\right)\right\}e_2^2-\frac{1}{2}\left(k+2\alpha_1\right)e_0e_1-\frac{1}{2}\left(1-k+2\alpha_2\right)e_0e_2 \\ +\frac{1}{2}\left\{\alpha_1(1-k)+\alpha_2k+2\alpha_1\alpha_2\right\}e_1e_2 \end{bmatrix}. \quad (4)$$

The bias of $\bar{y}_{G(RSS)}$, is given by

$$Bias\left(\bar{y}_{G(RSS)}\right)\cong\bar{Y}\begin{bmatrix} \frac{1}{8}\left\{k\left(4\alpha_1+3\right)+4\alpha_1\left(\alpha_1+1\right)\right\}V_{020}+\frac{1}{8}\left\{(1-k)\left(4\alpha_2+1\right)+4\alpha_2\left(\alpha_2+1\right)\right\}V_{002} \\ -\frac{1}{2}\left(k+2\alpha_1\right)V_{110}-\frac{1}{2}\left(1-k+2\alpha_2\right)V_{101}+\frac{1}{2}\left\{\alpha_1(1-k)+\alpha_2k+2\alpha_1\alpha_2\right\}V_{011} \end{bmatrix}. \quad (5)$$

Taking square of Eq. (4) and then expectations, the *MSE* of $\bar{y}_{G(RSS)}$, is given by

$$MSE\left(\bar{y}_{G(RSS)}\right)\cong\bar{Y}^2\begin{bmatrix} V_{200}+\frac{1}{4}\left(k+2\alpha_1\right)^2V_{020}+\frac{1}{4}\left(1-k+2\alpha_2\right)^2V_{002}-\left(k+2\alpha_1\right)V_{110} \\ -\left(1-k+2\alpha_2\right)V_{101}+\frac{1}{2}\left(k+2\alpha_1\right)\left(1-k+2\alpha_2\right)V_{011} \end{bmatrix}. \quad (6)$$

The optimum values of α_1 and α_2 are

$$\alpha_{1(opt)}=\frac{\left[2C_y(\rho_{yx}-\rho_{xz}\rho_{yz})-kC_x(1-\rho_{xz}^2)\right]}{2C_x(1-\rho_{xz}^2)} \quad (7)$$

and

$$\alpha_{2(opt)}=\frac{\left[2C_y(\rho_{yx}-\rho_{xz}\rho_{yz})-(1-k)C_z(1-\rho_{xz}^2)\right]}{2C_x(1-\rho_{xz}^2)}. \quad (8)$$

It is remarked that for different values of α_1 and α_2 in Eq (3), we can get various exponential ratio-type estimators from the proposed family of estimators $\bar{y}_{G(RSS)}$. Some are given below as:

(i) For $\alpha_1 = 1, \alpha_2 = 0$ and $k = 0$, we get

$$\bar{y}_{1(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{X}}{\bar{x}_{(rss)}}\right)\exp\left(\frac{\bar{Z}-\bar{z}_{[rss]}}{\bar{Z}+\bar{z}_{[rss]}}\right) \tag{9}$$

The bias and *MSE* of $\bar{y}_{1(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{1(RSS)}\right) \cong \bar{Y}\left[V_{020}+\frac{1}{8}V_{002}-V_{110}-\frac{1}{2}V_{101}+\frac{1}{2}V_{011}\right] \tag{10}$$

and

$$MSE\left(\bar{y}_{1(RSS)}\right) \cong \bar{Y}^2\left[V_{020}+V_{020}+\frac{1}{4}V_{002}-2V_{110}-V_{101}+V_{011}\right]. \tag{11}$$

(ii) For $\alpha_1 = 0$, $\alpha_2 = 1$ and $k = 1$, we get

$$\bar{y}_{2(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{Z}}{\bar{z}_{[rss]}}\right)\exp\left(\frac{\bar{X}-\bar{x}_{(rss)}}{\bar{X}+\bar{x}_{(rss)}}\right) \tag{12}$$

The bias and *MSE* of $\bar{y}_{2(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{2(RSS)}\right) \cong \bar{Y}\left[\frac{3}{8}V_{020}+V_{002}-\frac{1}{2}V_{110}-\frac{1}{2}V_{101}+V_{011}\right] \tag{13}$$

and

$$MSE\left(\bar{y}_{2(RSS)}\right) \cong \bar{Y}^2\left[V_{200}+\frac{1}{4}V_{020}+V_{002}-V_{110}-2V_{101}+V_{011}\right]. \tag{14}$$

(iii) For $\alpha_1 = 1, \alpha_2 = 0$ and $k = 1$, we get

$$\bar{y}_{3(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{X}}{\bar{x}_{(rss)}}\right)\exp\left(\frac{\bar{X}-\bar{x}_{(rss)}}{\bar{X}+\bar{x}_{(rss)}}\right) \tag{15}$$

The bias and *MSE* of $\bar{y}_{3(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{3(RSS)}\right) \cong \bar{Y}\left[\frac{15}{8}V_{020}-\frac{3}{2}V_{110}\right] \tag{16}$$

and

$$MSE\left(\bar{y}_{3(RSS)}\right) \cong \bar{Y}^2\left[V_{200}+\frac{9}{4}V_{020}-3V_{110}\right]. \tag{17}$$

(iv) For $\alpha_1 = 0, \alpha_2 = 1$ and $k = 0$, we get

$$\bar{y}_{4(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{Z}}{\bar{z}_{[rss]}}\right)\exp\left(\frac{\bar{Z}-\bar{z}_{[rss]}}{\bar{Z}+\bar{z}_{[rss]}}\right) \tag{18}$$

The bias and *MSE* of $\bar{y}_{4(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{4(RSS)}\right) \cong \bar{Y}\left[\frac{13}{8}V_{002} - \frac{3}{2}V_{101}\right] \tag{19}$$

and

$$MSE\left(\bar{y}_{4(RSS)}\right) \cong \bar{Y}^2\left[V_{200} + \frac{9}{4}V_{002} - 3V_{101}\right]. \tag{20}$$

(v) For $k=1$, Eq. (3) becomes

$$\bar{y}_{5(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{X}}{\bar{x}_{(rss)}}\right)^{\alpha_1}\left(\frac{\bar{Z}}{\bar{z}_{[rss]}}\right)^{\alpha_2}\exp\left(\frac{\bar{X} - \bar{x}_{(rss)}}{\bar{X} + \bar{x}_{(rss)}}\right) \tag{21}$$

The bias and *MSE* of $\bar{y}_{5(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{5(RSS)}\right) \approx \bar{Y}\left[\begin{array}{l}\frac{1}{8}\left(4\alpha_1^2 + 8\alpha_1 + 3\right)V_{020} + \frac{1}{2}\alpha_2\left(\alpha_2 + 1\right)V_{002} \\ -\frac{1}{2}\left(1 + 2\alpha_1\right)V_{101} - \alpha_2 V_{101} + \frac{1}{2}\alpha_2\left(1 + 2\alpha_1\right)V_{011}\end{array}\right] \tag{22}$$

and

$$MSE\left(\bar{y}_{5(RSS)}\right) \cong \bar{Y}^2\left[\begin{array}{l}V_{200} + \frac{1}{4}\left(1 + 2\alpha_1\right)^2 V_{020} + \alpha_2^2 V_{002} - \left(1 + 2\alpha_1\right)V_{110} \\ -2\alpha_2 V_{101} + \alpha_2\left(1 + 2\alpha_1\right)V_{011}\end{array}\right]. \tag{23}$$

The optimum values of α_1 and α_2 are

$$\alpha^*_{1(opt)} = \frac{2C_y\left(\rho_{yx} - \rho_{xz}\rho_{yz}\right) - C_x\left(1 - \rho_{xz}^2\right)}{2C_x\left(1 - \rho_{xz}^2\right)} \tag{24}$$

and

$$\alpha^*_{2(opt)} = \frac{C_y\left(\rho_{yx} - \rho_{xz}\rho_{yz}\right)}{2C_z\left(1 - \rho_{xz}^2\right)}. \tag{25}$$

By putting the optimum values of α_1 and α_2 in Eq. (22) and Eq. (23), we get the minimum bias and *MSE* of $\bar{y}_{5(RSS)}$ respectively as

$$Bias\left(\bar{y}_{5(RSS)}\right)_{\min} \cong \frac{1}{8}\bar{Y}\left(\gamma C_x^2 - W_{(x)}^2\right) \tag{26}$$

and

$$MSE\left(\bar{Y}_{5(RSS)}\right)_{\min} \cong \bar{Y}^2 V_{200}\left[1 - \frac{C_y^2}{C_x^2(1 - \rho_{xz}^2)}\left(\rho_{yx} - \rho_{xz}\right)^2\right]. \tag{27}$$

(vi) For $k=0$, Eq.(3) becomes

$$\bar{y}_{6(RSS)} = \bar{y}_{[rss]}\left(\frac{\bar{X}}{\bar{x}_{(rss)}}\right)^{\alpha_1}\left(\frac{\bar{Z}}{\bar{z}_{[rss]}}\right)^{\alpha_2}\exp\left(\frac{\bar{Z}-\bar{z}_{[rss]}}{\bar{Z}+\bar{z}_{[rss]}}\right) \quad (28)$$

The bias and *MSE* of $\bar{y}_{6(RSS)}$, are given respectively

$$Bias(\bar{y}_{6(RSS)}) \cong \bar{Y}\begin{bmatrix}\frac{1}{2}\alpha_1(\alpha_1+1)V_{020}+\frac{1}{8}\left(4\alpha_2^2+8\alpha_2+1\right)V_{002} \\ -\alpha_1 V_{110}-\frac{1}{2}(1+2\alpha_2)V_{101}+\frac{1}{2}\alpha_1(1+2\alpha_2)V_{011}\end{bmatrix} \quad (29)$$

and

$$MSE\left(\bar{y}_{6(RSS)}\right) \cong \bar{Y}^2\begin{bmatrix}V_{200}+\alpha_1^2 V_{020}+\frac{1}{4}(1+2\alpha_2)^2 V_{002}-2\alpha_1 V_{110} \\ -(1+2\alpha_2)V_{101}+2\alpha_1\alpha_2 V_{011}\end{bmatrix}. \quad (30)$$

The optimum values of α_1 and α_2

$$\alpha_{1(opt)}^{**} = \frac{C_y\left(\rho_{yx}-\rho_{xz}\rho_{yz}\right)}{2C_x\left(1-\rho_{xz}^2\right)} \quad (31)$$

and

$$\alpha_{2(opt)}^{**} = \frac{2C_y\left(\rho_{yx}-\rho_{xz}\rho_{yz}\right)-C_z\left(1-\rho_{xz}^2\right)}{2C_z\left(1-\rho_{xz}^2\right)}. \quad (32)$$

By putting Eqs.(31) and (32) in Eqs.(29) and (30), the minimum bias and *MSE* of $\bar{y}_{6(RSS)}$, are given respectively

$$Bias\left(\bar{y}_{6(RSS)}\right)_{\min} \cong \frac{1}{8}\bar{Y}\left(\gamma C_z^2 - W_{[z]}^2\right) \quad (33)$$

and

$$MSE\left(\bar{Y}_{6(RSS)}\right)_{\min} \cong \bar{Y}^2 V_{200}\left[1-\frac{C_y^2}{C_z^2(1-\rho_{xz}^2)}\left(\rho_{yz}-\rho_{xz}\right)^2\right]. \quad (34)$$

4. Efficiency Comparison

We obtain the conditions under which the proposed estimators are more efficient than the usual *RSS* mean estimator.

(i) Comparison: By Eq.(2) and Eq.(11)

$MSE\left(\bar{y}_{1(RSS)}\right) < MSE\left(\bar{y}_{(RSS)}\right)$, if

$$\frac{V_{020}+\frac{1}{4}V_{002}}{2V_{110}+V_{101}-V_{011}} < 1$$

(ii) Comparison: By Eq.(2) and Eq.(14)

$$MSE\left(\bar{y}_{2(RSS)}\right) < MSE\left(\bar{y}_{(RSS)}\right), \text{ if}$$

$$\frac{\frac{1}{4}V_{020} + V_{002}}{V_{110} + 2V_{101} - V_{011}} < 1$$

(iii) Comparison: By Eq.(2) and Eq.(17)

$$MSE\left(\bar{y}_{3(RSS)}\right) < MSE\left(\bar{y}_{(RSS)}\right) \text{ if}$$

$$\frac{3C_x}{4C_y\rho_{yx}} < 1$$

(iv) Comparison: By Eq.(2) and Eq.(20)

$$MSE\left(\bar{y}_{4(RSS)}\right) < MSE\left(\bar{y}_{(RSS)}\right), \text{ if}$$

$$\frac{3C_z}{4C_y\rho_{yz}} < 1$$

(v) Comparison: By Eq.(2) and Eq.(27)

$$MSE\left(\bar{y}_{5(RSS)}\right)_{\min} < MSE\left(\bar{y}_{(RSS)}\right), \text{ if}$$

$$\left(\rho_{yx} - \rho_{xz}\right)^2 > 0$$

(vi) Comparison: By Eq.(2) and Eq.(34)

$$MSE\left(\bar{y}_{6(RSS)}\right)_{\min} < MSE\left(\bar{y}_{(RSS)}\right), \text{ if}$$

$$\left(\rho_{yz} - \rho_{xz}\right)^2 > 0$$

5. Simulation Study

To obtain *MSE*, Relative Bias (*RB*) and Relative Root Mean Square Error (*RRMSE*) of the proposed class of exponential ratio-type estimators, a simulation study is conducted. Ranking is performed on basis of the concomitant variable X. Trivariate random observation $\left(X, Y, Z\right)$ are generated from a trivariate Gamma distribution with known population correlation coefficients $\rho_{yx} = 0.90$, $\rho_{yz} = 0.80$ and $\rho_{xz} = 0.70$. Using 20,000 simulations, estimates of *MSE*, *RRMSE* and *RB* for different estimators are computed using ranked set sampling scheme as described in Section 2. Estimators are then compared in the term of *MSE, RB, RRMSE* and percentage relative efficiency (*PRE*). We have computed the *PRE* of different ratio-in-exponential type estimators of population mean $\left(\bar{Y}\right)$ with respect to usual unbiased mean estimator $\bar{y}_{(RSS)}$ for different values of m

and r. The results are shown in Tables 1, 2, 3 and 4. The findings indicate that with increase in sample size, *MSEs*, *RB*, *RRMSEs* decrease which are expected results. We used the following expressions to obtain the *MSE*, *RB*, *RRMSE* and *PRE*:

$$RB\left(\bar{y}_{G(RSS)}\right)=\frac{1}{\bar{Y}}\left[\frac{1}{20000}\sum_{i=1}^{20000}(\bar{y}_{G(RSS)i}-\bar{Y})\right] \quad , \qquad G=1,2,\ldots,6$$

$$RRMSE\left(\bar{y}_{G(RSS)}\right)=\frac{1}{\bar{Y}}\left[\frac{1}{20000}\sum_{i=1}^{20000}(\bar{y}_{G(RSS)i}-\bar{Y})^2\right]^{\frac{1}{2}} \ ,$$

$$MSE\left(\bar{y}_{G(RSS)}\right)=\frac{1}{20000}\sum_{i=1}^{20000}(\bar{y}_{G(RSS)i}-\bar{Y})^2 \text{ and}$$

$$PRE=\frac{MSE\left(\bar{y}_{(RSS)}\right)}{MSE\left(\bar{y}_{G(RSS)}\right)}\times 100 \ , \quad G=1,2,\ldots,6$$

Table 1: The Simulated *MSE* of Different Estimators

m	r	n	$\bar{y}_{(RSS)}$	$\bar{y}_{1(RSS)}$	$\bar{y}_{2(RSS)}$	$\bar{y}_{3(RSS)}$	$\bar{y}_{4(RSS)}$	$\bar{y}_{5(RSS)}$	$\bar{y}_{6(RSS)}$
	3	9	0.18346	0.08961	0.09566	0.11084	0.12941	0.03522	0.03555
3	4	12	0.13698	0.06838	0.07344	0.08287	0.09813	0.02641	0.02658
	5	15	0.10966	0.05705	0.06017	0.06947	0.07879	0.02232	0.02242
	10	30	0.05356	0.02673	0.02873	0.03234	0.03831	0.01059	0.01074
	15	45	0.03691	0.01742	0.01899	0.02052	0.02523	0.00681	0.00695
	20	60	0.02675	0.01282	0.01357	0.01548	0.01771	0.00519	0.00527
	3	12	0.11585	0.06300	0.06884	0.07584	0.09344	0.02714	0.02721
4	4	16	0.09017	0.04688	0.05179	0.05550	0.07036	0.01984	0.01996
	5	20	0.06775	0.03629	0.03940	0.04408	0.05339	0.01623	0.01634
	10	40	0.03479	0.01716	0.01885	0.02091	0.02596	0.00791	0.00800
	15	60	0.02334	0.01151	0.01247	0.01397	0.01685	0.005293	0.00529
	20	80	0.01790	0.00914	0.00996	0.01082	0.01328	0.00408	0.00410
	3	15	0.08581	0.04440	0.04998	0.05344	0.07011	0.02200	0.02201
	4	20	0.06093	0.03386	0.03789	0.04082	0.05294	0.01601	0.01605
5	5	25	0.05037	0.02683	0.02881	0.03340	0.03936	0.01287	0.01292
	10	50	0.02505	0.01339	0.01462	0.016303	0.01997	0.00626	0.00629
	15	75	0.01655	0.00873	0.00963	0.01052	0.01323	0.00415	0.00418
	20	100	0.01218	0.00667	0.00732	0.00805	0.01005	0.00311	0.00316

Table 2: The Percentage *RE* of Proposed Estimators with respect $\bar{y}_{(RSS)}$

m	r	n	$\bar{y}_{(RSS)}$	$\bar{y}_{1(RSS)}$	$\bar{y}_{2(RSS)}$	$\bar{y}_{3(RSS)}$	$\bar{y}_{4(RSS)}$	$\bar{y}_{5(RSS)}$	$\bar{y}_{6(RSS)}$
	3	9	100	204.73	191.78	165.50	141.76	516.55	516.01
3	4	12	100	200.30	186.50	165.28	139.59	515.24	515.32
	5	15	100	192.21	182.24	157.84	139.18	489.10	589.12
	10	30	100	200.33	186.39	165.60	139.80	498.52	598.53
	15	45	100	211.80	194.30	179.87	146.28	529.05	529.03
	20	60	100	208.68	197.12	172.81	151.01	507.06	507.01
	3	12	100	183.87	168.27	152.75	123.97	425.82	425.65
4	4	16	100	192.34	174.09	162.46	128.16	451.54	451.60
	5	20	100	186.64	171.94	153.68	126.88	414.77	414.76
	10	40	100	202.65	184.53	184.52	166.46	434.75	434.78
	15	60	100	202.83	187.12	166.99	138.47	440.97	440.99
	20	80	100	195.82	179.70	165.42	134.72	435.69	435.70
	3	15	100	193.26	171.67	160.58	122.39	389.70	389.75
	4	20	100	179.95	160.80	149.25	115.10	380.59	380.64
5	5	25	100	187.71	174.78	150.79	127.96	391.32	391.25
	10	50	100	186.99	171.41	153.68	125.42	398.21	398.16
	15	75	100	189.63	171.92	157.40	125.15	396.04	396.03
	20	100	100	182.64	166.21	151.14	121.25	385.14	385.13

Table 3: The Simulated Percentage *RRMSE* of Different Estimators

m	r	n	$\bar{y}_{(RSS)}$	$\bar{y}_{1(RSS)}$	$\bar{y}_{2(RSS)}$	$\bar{y}_{3(RSS)}$	$\bar{y}_{4(RSS)}$	$\bar{y}_{5(RSS)}$	$\bar{y}_{6(RSS)}$
	3	9	21.69	15.25	15.79	16.96	18.39	9.52	9.54
3	4	12	18.74	13.38	13.92	14.69	16.11	8.20	8.23
	5	15	16.77	12.16	12.50	13.41	14.32	7.53	7.58
	10	30	11.72	8.30	8.61	9.12	9.95	5.19	5.21
	15	45	9.73	6.68	6.98	7.25	8.05	4.21	4.23
	20	60	8.28	5.74	5.90	6.31	6.74	3.66	3.67
	3	12	17.25	12.84	13.43	14.08	15.65	8.33	8.35
4	4	16	15.20	11.05	11.63	12.01	13.57	7.11	7.15
	5	20	13.18	9.68	10.09	10.67	11.76	6.40	6.43
	10	40	9.44	6.65	6.97	7.33	8.18	4.53	4.56
	15	60	7.74	5.44	5.66	6.00	6.58	3.64	3.68
	20	80	6.77	4.85	5.06	5.27	5.85	3.20	3.23
	3	15	14.83	10.70	11.36	11.73	13.47	7.49	7.47
	4	20	12.50	9.35	9.89	10.27	11.69	6.40	6.42
5	5	25	11.37	8.32	8.62	9.29	10.08	5.71	5.73
	10	50	8.01	5.86	6.13	6.47	7.17	4.01	4.03
	15	75	6.51	4.73	4.97	5.19	5.83	3.27	3.28
	20	100	5.58	4.13	4.33	4.54	5.07	2.84	2.86

Table 4: The Simulated Percentage *RB* of Different Estimators

m	r	n	$\bar{y}_{(RSS)}$	$\bar{y}_{1(RSS)}$	$\bar{y}_{2(RSS)}$	$\bar{y}_{3(RSS)}$	$\bar{y}_{4(RSS)}$	$\bar{y}_{5(RSS)}$	$\bar{y}_{6(RSS)}$
	3	9	0.20	1.99	1.73	1.81	1.56	0.51	0.52
3	4	12	0.14	1.62	1.12	1.56	1.82	0.18	0.16
	5	15	0.08	1.43	1.30	1.43	1.76	0.08	0.07
	10	30	-0.07	0.68	0.56	0.68	0.85	-0.05	-0.04
	15	45	-0.05	0.29	0.20	0.29	0.45	-0.03	-0.03
	20	60	-0.02	0.23	0.26	0.23	0.24	-0.01	-0.01
	3	12	-0.76	1.99	1.86	1.99	1.33	0.20	0.19
4	4	16	-0.03	1.58	1.41	1.58	1.92	0.36	0.35
	5	20	0.09	0.95	0.88	0.95	1.16	0.20	0.18
	10	40	-0.02	0.53	0.48	0.53	0.65	0.10	0.11
	15	60	-0.04	0.26	0.31	0.26	0.25	0.05	0.05
	20	80	-0.02	0.35	0.28	0.35	0.44	0.04	0.03
	3	15	0.24	0.95	0.80	0.95	0.83	0.06	0.06
5	4	20	-0.21	0.83	0.82	0.82	0.96	0.07	0.05
	5	25	-0.17	0.71	0.73	0.71	0.79	0.14	0.13
	10	50	-0.09	0.32	0.27	0.32	0.41	-0.02	-0.02
	15	75	0.02	0.23	0.20	0.23	0.29	0.03	0.01
	20	100	-0.01	0.12	0.11	0.12	0.16	0.00	0.00

6. Conclusion

In Tables 1 and 3, we see that the proposed ratio-in-exponential type estimators $\bar{y}_{G(RSS)}$, have less *MSE* and *RRMSE* values as compared to $\bar{y}_{(RSS)}$. Also, *MSE* and *RRMSE* decrease with increase in the sample size. The simulation result of Table 4 indicate that the proposed estimators have reasonable biases, since the values of percentage *RB* are all less than 2% in absolute terms. Also, the value of percentage *RB* decreases with increase in sample size $n = mr$. So, we conclude that the proposed ratio-in-exponential type estimators are preferable than the usual mean estimator under *RSS* scheme.

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