

A comparative study to estimate the parameters of mixed-Weibull distribution

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Abstract

A mixture of Weibull distribution, involving two shape parameters, two scale parameters, and one proportionality parameter, is simulated through the statistical software MINITAB (1996). The two methods Graphical (John H.K. Kao 1959) and moments based (Lee W. Fall 1970) are applied to estimate the underlying five parameters of mixed Weibull distribution in order to find the accuracy and precision of the estimator.

1. Introduction

The Weibull distribution, derived in 1939 by W. Weibull has been recognized as an appropriate model in reliability studies and life-testing problems such as time to failure or life length of a component or a product.

Distributions resulting from mixing two or more component distributions are designated as 'mixed' or 'compound'. Some methods for obtaining efficient estimates of the mixed Weibull distribution have been outlined in recent years. A method for estimating parameters of mixed distributions using sample moments has been outlined by Paul R. Rider (1961) who considered compound Poisson, Binomial, and a special case of the mixed Weibull distribution. A graphical method for estimating the mixed Weibull parameters in life testing of electron tubes is proposed by John H. K. Kao (1959).

This paper presents a comparison of the estimation methods to find the accuracy and precision of the estimator. For this purpose we simulate the data from Weibull distribution using MINITAB (1996) and apply the two methods of estimating the parameters of mixed Weibull distribution.

1.1 Simple Weibull distribution

A simple Weibull probability distribution function is defined as

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right] ; x > 0$$

Where $\beta > 0$ and $\alpha > 0$ are the parameters referred to as shape and scale parameter of the distribution.

The strength of the Weibull lies in its flexible shape as a model for many different kinds of data. The parameter β plays the role in determining, how the Weibull will look.

Table1: Weibull Distribution Properties

Shape Parameter	PDF
$0 < \beta < 1$	Exponentially decay from infinity
$\beta = 1$	Exponentially decay from 1/mean
$\beta > 1$	Rises to peak and then decrease
$\beta = 2$	Reyleigh distribution
$3 \leq \beta \leq 4$	Has normal bell shape appearance
$\beta > 10$	Has shape very similar to type 1 extreme value distribution

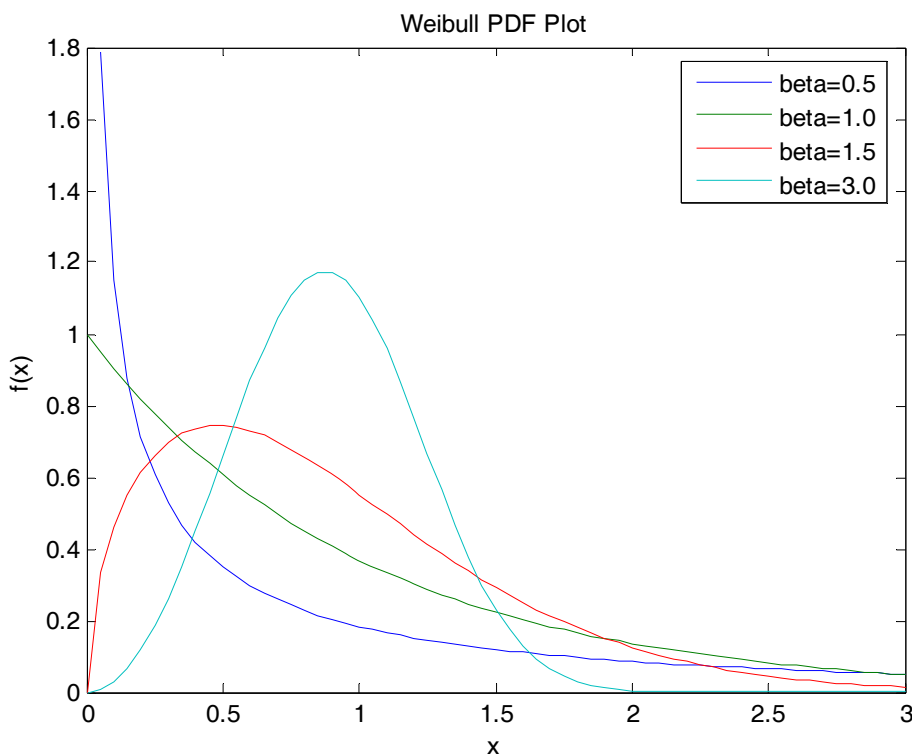


Figure1: Shows some of the simple Weibull distribution for $\beta = 0.5, 1, 1.5$ and 3 .

1.2 Mixed-Weibull Distribution

A k two-parametric Weibull probability distribution function is defined as

$$f(x) = \sum_{i=1}^k P_i f_i(x) \quad ; \quad \sum_{i=1}^k P_i = 1$$

Where;

$f_i(x) \Rightarrow$ *ith sub population*

$P_i \Rightarrow$ *proportion of sub – population (mixture parameter)*

A mixture distribution involving two-two parametric Weibull distribution is

$$f(x) = P_1 f_1(x) + P_2 f_2(x) \quad ; P_1 + P_2 = 1$$

$$f(x) = P f_1(x) + (1 - P) f_2(x) \tag{1}$$

$$f(x) = P \frac{\beta_1}{\alpha_1} \left(\frac{x}{\alpha_1}\right)^{\beta_1-1} \exp\left\{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}\right\} + (1 - P) \frac{\beta_2}{\alpha_2} \left(\frac{x}{\alpha_2}\right)^{\beta_2-1} \exp\left\{-\left(\frac{x}{\alpha_2}\right)^{\beta_2}\right\} \quad ; x > 0 \tag{2}$$

Where;

$\alpha > 0$, scale parameter, $\beta > 0$, shape parameter, $P > 0$, mixture parameter

CDF of mixed- Weibull distribution is defined as

$$F(x) = P F_1(x) + (1 - P) F_2(x) \tag{3}$$

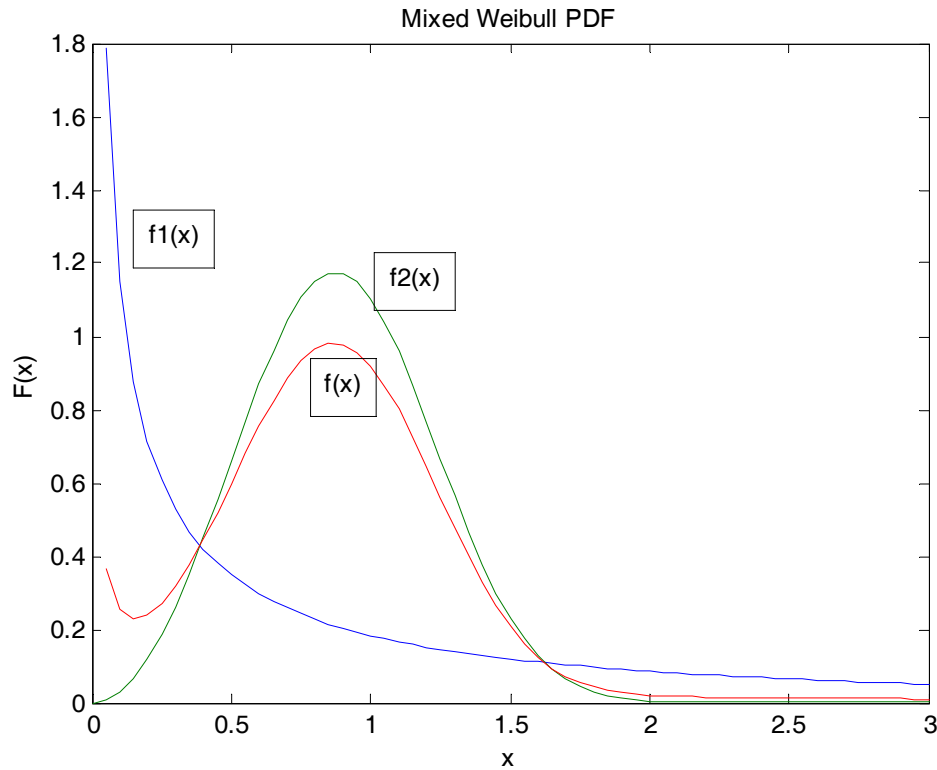


Figure 2: Mixed Weibull probability density function
 $(\alpha_1 = \alpha_2 = 1, p = 0.2, \beta_1 = 0.5 \text{ and } \beta_2 = 3)$

2. Estimation of the parameters of mixed-Weibull distribution

Numerous methods for obtaining efficient estimates of the simple Weibull distribution have been outlined in recent years; however we can not apply all these methods to the mixed-Weibull distribution due to complexity of the functional form. We therefore consider the two methods of estimation Graphical and Sample moments to find the accuracy and precision of the estimator.

2.1 Graphical Method

A graphical procedure for estimation of mixed-Weibull parameters in life-testing of electron tubes is given by John H. K. Kao (1959). This method of estimating the parameters is based upon the fact that a Weibull cumulative distribution becomes a straight line in \ln versus \ln - \ln coordinates.

1. Plot the sample cdf on Weibull probability paper and fit by inspection a smooth curve (called Weibull plot).
2. Starting from each end of the Weibull plot draw a tangent line and denote them by $p\hat{F}_1$ and $(1-p)\hat{F}_2$ which are the estimates of $pF_1(x)$ and $qF_2(x)$ respectively.
3. At the intersection of $(1-p)\hat{F}_2$ with the upper boarder line drop a vertical line whose intersection with $p\hat{F}_1$ as read from the percent failure scale gives \hat{p} the estimate of p and hence $1-p=q$.
4. Mixed Weibull distribution may now be separated into the two sub-populations. The estimated number of items belong to $F_1(x)$ and $F_2(x)$ are $n\hat{p}$ and $n\hat{q}$ respectively from which the sample cdf's of $F_1(x)$ and $F_2(x)$ may be calculated.
5. The y-intercept and slope of the plots of the separated sub-populations $F_i(x)$, give the usual estimates of $-\beta \ln \alpha_i$ and β_i respectively.

2.2 Method of sample moments

The r th theoretical moment about origin of the mixed Weibull distribution $f(x)$ is

$$\mu'_r = p \int_0^{\infty} x^r f_1(x) dx + (1-p) \int_0^{\infty} x^r f_2(x) dx \quad (4)$$

where $f_i(x)$ are defined as in equation (2).

The first five theoretical moments about origin follow as

$$\mu'_r = p \alpha_1^r \Gamma\left(\frac{r}{\beta_1} + 1\right) + (1-p) \alpha_2^r \Gamma\left(\frac{r}{\beta_2} + 1\right) \quad \text{for } r = 1, 2, 3, 4, 5 \quad (5)$$

Now equating the theoretical moments to the corresponding sample moments, the set of equations (5) becomes

$$m_r = p \alpha_1^r \Gamma\left(\frac{r}{\beta_1} + 1\right) + (1-p) \alpha_2^r \Gamma\left(\frac{r}{\beta_2} + 1\right) \quad \text{for } r = 1, 2, 3, 4, 5 \quad (6)$$

Where m_r is the sample moment about the origin.

The set of equations (6) is a system of five equations which must be solved simultaneously for estimates of the five parameters $p, \alpha_1, \alpha_2, \beta_1, \beta_2$. For convenience in handling these equations, we will make the following transformations where necessary in this paper.

Let

$$\begin{aligned} u_1 &= \Gamma\left(\frac{1}{\beta_1} + 1\right) & v_1 &= \Gamma\left(\frac{1}{\beta_2} + 1\right) \\ u_2 &= \Gamma\left(\frac{2}{\beta_1} + 1\right) & v_2 &= \Gamma\left(\frac{2}{\beta_2} + 1\right) \\ u_3 &= \Gamma\left(\frac{3}{\beta_1} + 1\right) & v_3 &= \Gamma\left(\frac{3}{\beta_2} + 1\right) \\ u_4 &= \Gamma\left(\frac{4}{\beta_1} + 1\right) & v_4 &= \Gamma\left(\frac{4}{\beta_2} + 1\right) \end{aligned}$$

Thus, the system of equations (6) becomes

$$m_r = p\alpha_1^r u_r + (1-p)\alpha_2^r v_r \quad r=1, 2, 3, 4, 5 \quad (7)$$

Solving equations (7) for $r=1, 2$ to obtain the estimates of α_1 and α_2 , yields

$$\hat{\alpha}_2 = \frac{m_1(1-p)u_2v_1 \pm \sqrt{-p(1-p)m_1^2v_2u_1^2u_2 + p(1-p)^2m_2u_1^2u_2v_1^2 + p^2(1-p)m^2u_1^4v_2}}{(1-p)^2u_2v_1^2 + p(1-p)v_2u_1^2} \quad (8)$$

$$\hat{\alpha}_1 = \frac{m_1}{pu_1} - \frac{(1-p)v_1}{pu_1} \left[\frac{m_1(1-p)u_2v_1 \pm \sqrt{-p(1-p)m_1^2v_2u_1^2u_2 + p(1-p)^2m_2u_1^2u_2v_1^2 + p^2(1-p)m^2u_1^4v_2}}{(1-p)^2u_2v_1^2 + p(1-p)v_2u_1^2} \right] \quad (9)$$

Substituting the expression of α_1 and α_2 in the next two equations, obtained from the system of equations (7) for $r = 3$ and 4, there are now two equations and three unknowns. At this point, the explicit expressions for the unknown parameters p, β_1, β_2 can not be obtained. Therefore the graphical method is used to obtain the estimate of p , the proportionality parameter. Once an estimate of p has been determined graphically, solve the two equations for β_1 , and β_2 by the Newton–Raphson method for solving non-linear system of equations (NRSYS), defined as follows

$$x_{k+1} = x_k + dx_k \quad \text{where } dx_k = -f'(x_k)^{-1} f(x_k)$$

The vector form of the non-linear system $f(\beta) = 0$

Where;

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad f(\beta) = \begin{bmatrix} f_1(\beta_1, \beta_2) \\ f_2(\beta_1, \beta_2) \end{bmatrix} \quad f'(\beta) = \begin{bmatrix} \frac{\partial}{\partial \beta_1} f_1(\beta_1, \beta_2) & \frac{\partial}{\partial \beta_2} f_1(\beta_1, \beta_2) \\ \frac{\partial}{\partial \beta_1} f_2(\beta_1, \beta_2) & \frac{\partial}{\partial \beta_2} f_2(\beta_1, \beta_2) \end{bmatrix}$$

Hence,

$$\beta_{k+1} = \beta_k - f'(\beta_k)^{-1} f(\beta_k) \tag{10}$$

Once β_1 , and β_2 have been determined by the equation (10), estimates of α_1 and α_2 are obtained from equation (8) and (9) respectively.

3. Simulation from MINITAB

We simulate a random sample of size 100 in such a way that in the sample first 40 values have parametric values $\beta_1=0.5$ and $\alpha_1=1$, and last 60 values have parametric values $\beta_2=2.5$ and $\alpha_2=10$. The proportion of first 40 values is $40/100 = 0.4$ and hence $1-p = q = 0.6$.

3.1 Estimation by Graphical Method

Employing the graphical technique described in session 2.1, provides an estimate of p , the proportionality parameter, and equal to 0.39 as shown in Figure 3. The estimated number of items belongs to $F_1(x)$ and $F_2(x)$ are 39 and 61 respectively.

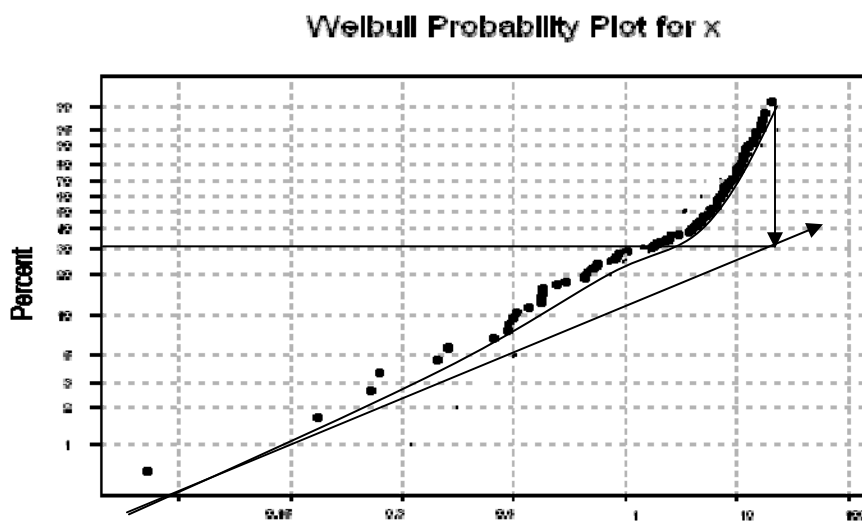


Figure3: Welbull Plot

Figure 4 shows the Weibull cumulative plots of the separated sub-population $F_i(x)$. The y-intercepts and slopes of the two straight lines are determined and hence the estimates of the remaining parameters are

α_1	α_2	β_1	β_2
0.85	10.7	0.47	2.35

3.2 Estimation by Method of moments

The iterations from the set of equations (8), (9), and (10) are obtained, as follow

Table 2: Iteration values

β_1	β_2	α_1	α_2
0.5	1	1.393712	8.362272
0.38687	1.47451	0.315633	10.39815
0.38068	2.14614	0.243961	10.77967
0.41655	3.17987	0.367359	10.54711
0.48625	1.89023	0.865206	10.11597
0.52899	2.56534	1.090066	10.0033

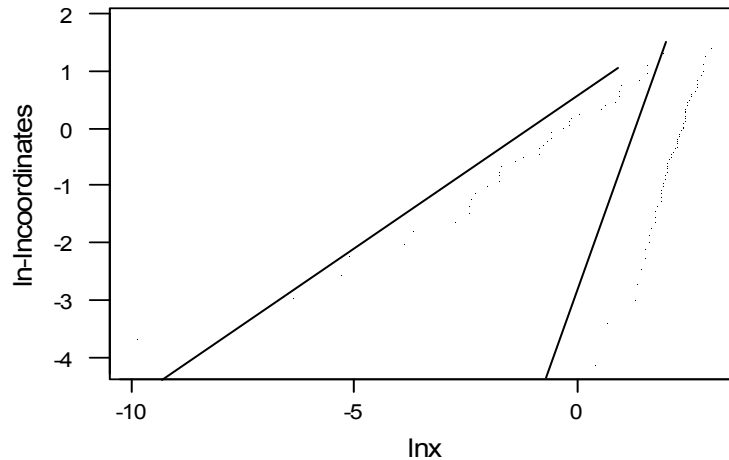


Figure 4: In versus In-In coordinates

4. Conclusion

By comparing the sampling errors of three methods of estimation, the error of estimation are least by the method of sample moments for mixture of two Weibull distributions as can be seen in Table 3.

Table 3: Summarizes the results obtained by all concerned methods

Parameters		Graphical		Moments	
		Estimates	Sampling error	Estimates	Sampling error
P	0.4	0.39	0.01	0.39 (graphical)	0.01
α_1	1	0.85	0.15	1.09	0.09
α_2	10	10.7	0.7	10.0	0
β_1	0.5	0.47	0.03	0.52	0.02
β_2	2.5	2.35	0.15	2.56	0.06

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