

The Kumaraswamy–Inverse Weibull Distribution

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Abstract

In this paper we have proposed a new four parameter Inverse Weibull distribution that is based upon the cumulative distribution function of Kumaraswamy (1980) distribution. The distributional properties of the proposed distribution have been studied. Special cases of the proposed distribution have also been explored.

Keywords: Kumaraswamy Distribution, Inverse Weibull Distribution, Generalized Distributions.

1. Introduction

The Beta distribution with density function:

$$f_B(x) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1}; \quad 0 < x < 1, (a,b) \in \mathfrak{R}^+ \quad (1.1)$$

is perhaps one of the most popular bounded continuous probability distribution. This distribution has attracted lot of attention within the area of theoretical and applied statistics. The distribution has found extensive application in areas including; but not limited to; engineering, computer sciences and hydrology. Eugene, Lee and Famoye (2002) have used the Beta distribution to generalize the Normal distribution. Eugene et al. (2002) has shown that the logit of Beta distribution can be used to generalize any parent cumulative distribution function $G(x)$ as:

$$f_{B-G}(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1} (1-w)^{b-1} dw; \quad (1.2)$$

where $I_x(a,b)$ is incomplete Beta function ratio given as $I_x(a,b) = B_x(a,b)/B(a,b)$.

Eugene et al. (2002) has used the CDF of Normal distribution in (1.2) to propose the Beta–Normal distribution. The generalization given in (1.2) has been used by number of authors to propose new distributions. Some notable references include Nadarajah and Kotz (2004, 2005), Famoye, Lee and Olumolade (2005), Hanook, Shahbaz, Mohsin and Kibria (2012) and many others.

Although the generalization of distribution functions given in (1.2) has attracted number of researchers, it still involves the complexity of incomplete beta function ratio. Some researchers have suggested to use other bounded distributions on (0,1) to obtain the generalization of any parent cumulative distribution function. One such distribution is the Kumaraswamy (1980) distribution having density and distribution function as:

$$f_K(x) = abw^{a-1}(1-w^a)^{b-1}; 0 < x < 1, (a, b) \in \mathfrak{R}^+; \quad (1.3)$$

and $F_K(x) = 1 - (1 - x^a)^b. \quad (1.4)$

Cordeiro and Castro (2010) have proposed a generalization of probability distributions by using (1.3) and (1.4). The density and distribution function of generalized class given by Cordeiro and Castro (2010) has following forms:

$$f_{K-G}(x) = abg(x)\{G(x)\}^{a-1}\left[1 - \{G(x)\}^a\right]^{b-1}; \quad (1.5)$$

and $F_{K-G}(x) = 1 - \left[1 - \{G(x)\}^a\right]^b. \quad (1.6)$

Cordeiro, Ortega and Nadarajah (2010) have used the CDF of Weibull distribution in (1.5) to propose the Kumaraswamy–Weibull distribution. The generalization (1.5) and (1.6) can be used to propose other distributions based on the Kumaraswamy distribution.

In the following we have proposed the Kumaraswamy–Inverse Weibull (*Kum–IW* for short) distribution by using the CDF of Inverse Weibull distribution in (1.5).

2. The Kumaraswamy–Inverse Weibull Distribution

The Inverse Weibull distribution has been used as an effective model for failure data in the literature. The density and distribution function of Inverse Weibull random variable is given as:

$$f_{IW}(x) = \frac{\alpha\beta}{x^{\beta+1}} \exp\left(-\frac{\alpha}{x^\beta}\right); (x, \alpha, \beta) \in \mathfrak{R}^+; \quad (2.1)$$

and $F_{IW}(x) = \exp\left(-\frac{\alpha}{x^\beta}\right). \quad (2.2)$

The distribution (2.1) has been used by Hanook et al. (2012) to propose the Beta Inverse Weibull. We propose the Kumaraswamy–Inverse Weibull distribution by using the distribution function (2.2) in (1.5). The density function of Kum–IW distribution is given as:

$$f_{K-IW}(x) = ab \frac{\alpha\beta}{x^{\beta+1}} \exp\left(-\frac{\alpha}{x^\beta}\right) \left\{ \exp\left(-\frac{\alpha}{x^\beta}\right) \right\}^{a-1} \left[1 - \left\{ \exp\left(-\frac{\alpha}{x^\beta}\right) \right\}^a \right]^{b-1}$$

or
$$f_{K-IW}(x) = \frac{ab\alpha\beta}{x^{\beta+1}} \exp\left(-\frac{a\alpha}{x^\beta}\right) \left[1 - \exp\left(-\frac{a\alpha}{x^\beta}\right) \right]^{b-1}; (x, a, b, \alpha, \beta) \in \mathfrak{R}^+. \quad (2.3)$$

We will use the notation $KIW(a, b, \alpha, \beta)$ to denote the density (2.3). Using following series representation of Prudnikov *et al.* (1986):

$$(1+x)^\alpha = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{j! \Gamma(\alpha+1-j)} x^j;$$

the density (2.3) can be written as:

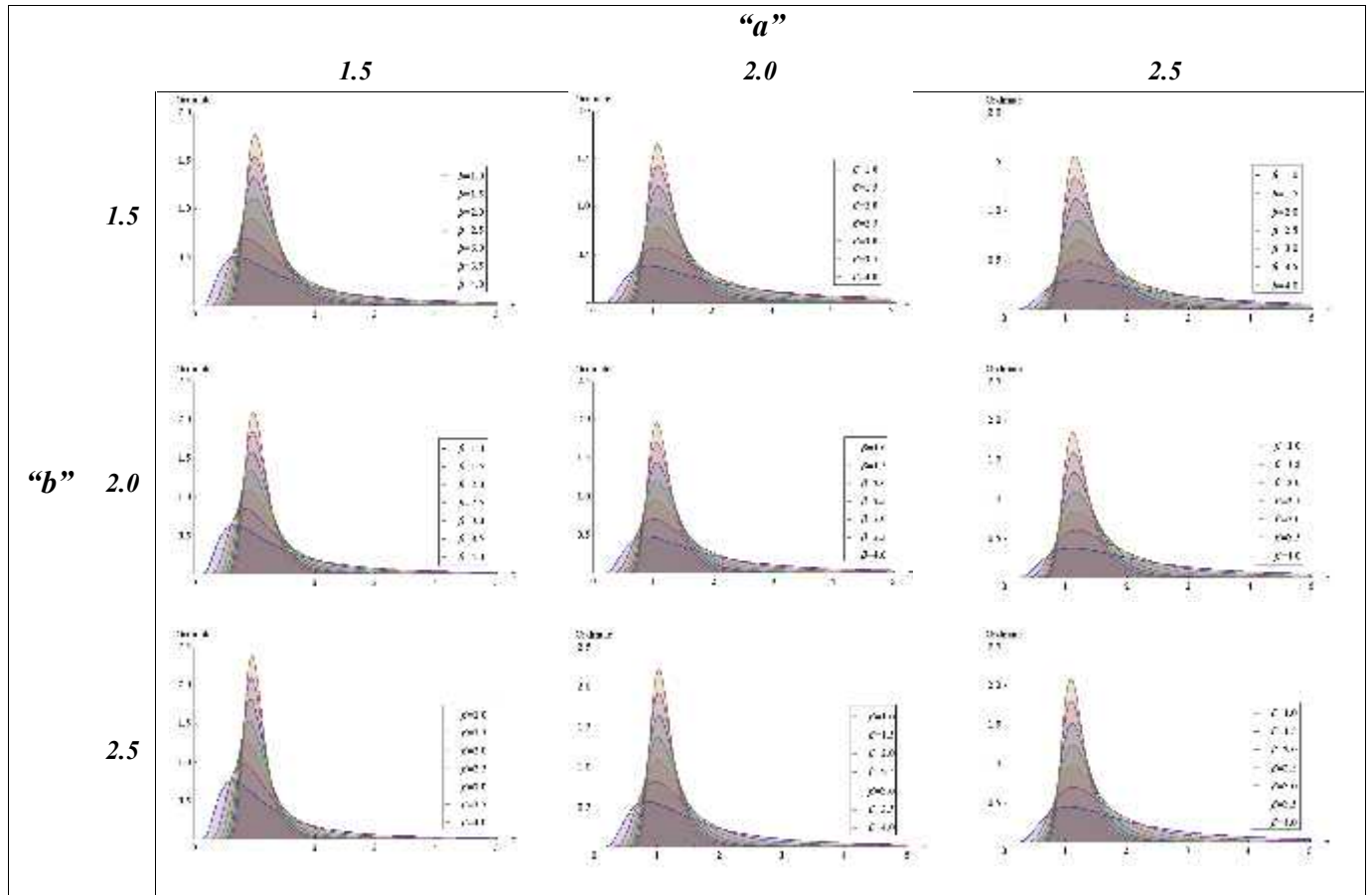
$$f_{K-IW} = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(b)}{j! \Gamma(b-j)} \cdot \frac{ab\alpha\beta}{x^{\beta+1}} \cdot \exp\left\{-\frac{a\alpha(j+1)}{x^\beta}\right\}. \quad (2.4)$$

The density (2.4) can be viewed as an infinite weighted sum of Inverse Weibull densities. The distribution function of Kum–IW distribution is immediately written by using (1.6) as:

$$F_{K-IW}(x) = 1 - \left[1 - \exp\left(-\frac{a\alpha}{x^\beta}\right) \right]^b. \quad (2.5)$$

The graphs of density are shown below for various choices of the parameters. The plot shows that the parameter β controls the kurtosis of the distribution.

Figure 1: The plot of $KIW(a, b, \alpha, \beta)$ for various choices of the parameters



3. Basic Properties of the Distribution

In this section we have studied the basic properties of Kum–IW distribution. We first evaluate the moments of the proposed distribution as under:

$$\begin{aligned}
 \mu'_r &= E(X^r) = \int_0^\infty x^r f_{K-IW}(x) dx \\
 &= \int_0^\infty x^r \sum_{j=0}^\infty (-1)^j \frac{\Gamma(b)}{j! \Gamma(b-j)} \cdot \frac{ab\alpha\beta}{x^{\beta+1}} \cdot \exp\left\{-\frac{a\alpha(j+1)}{x^\beta}\right\} dx \\
 &= \sum_{j=0}^\infty (-1)^j \frac{ab\alpha\beta\Gamma(b)}{j! \Gamma(b-j)} \int_0^\infty x^r \cdot \frac{1}{x^{\beta+1}} \cdot \exp\left\{-\frac{a\alpha(j+1)}{x^\beta}\right\} dx
 \end{aligned}$$

Making the transformation $w = a\alpha(j+1)/x^\beta$ in above expression and simplifying, the r -th moment of Kum–IW distribution is:

$$\mu'_r = \sum_{j=0}^\infty (-1)^j \frac{ab\alpha\beta\Gamma(b)\Gamma(1-r/\beta)}{j! \beta [a\alpha(j+1)]^{1-r/\beta} \Gamma(b-j)}; \tag{3.1}$$

which exist for $r < \beta$. The mean, variance, Skewness and Kurtosis can be obtained from (3.1). Numerical values of Mean, Variance, Skewness and Kurtosis for various choices of parameters are given in Table 3.1 through Table 3.3.

Table 3.1: Mean of Kum–IW distribution for various choices of parameters

a	b	β				
		2	3	4	5	6
1	1	1.7725	1.3541	1.2254	1.1642	1.1288
	2	1.0383	1.0022	0.9936	0.9911	0.9906
	3	0.8675	0.8971	0.9172	0.9310	0.9409
	4	0.7851	0.8424	0.8760	0.8979	0.9133
	5	0.7346	0.8074	0.8492	0.8761	0.8949
2	1	2.5066	1.7061	1.4573	1.3374	1.2670
	2	1.4684	1.2626	1.1816	1.1385	1.1119
	3	1.2268	1.1303	1.0907	1.0694	1.0561
	4	1.1102	1.0613	1.0417	1.0314	1.0251
	5	1.0389	1.0173	1.0098	1.0064	1.0045
3	1	3.0700	1.9530	1.6127	1.4503	1.3556
	2	1.7984	1.4454	1.3076	1.2347	1.1896
	3	1.5025	1.2938	1.2071	1.1597	1.1300
	4	1.3598	1.2149	1.1529	1.1185	1.0968
	5	1.2723	1.1645	1.1176	1.0914	1.0747
4	1	3.5449	2.1495	1.7330	1.5362	1.4222
	2	2.0766	1.5908	1.4051	1.3078	1.2480
	3	1.7349	1.4240	1.2971	1.2284	1.1855
	4	1.5701	1.3372	1.2388	1.1848	1.1506
	5	1.4692	1.2817	1.2009	1.1560	1.1275
5	1	3.9633	2.3155	1.8324	1.6063	1.4761
	2	2.3217	1.7137	1.4857	1.3675	1.2953
	3	1.9397	1.5340	1.3715	1.2845	1.2304
	4	1.7554	1.4404	1.3099	1.2388	1.1942
	5	1.6426	1.3806	1.2698	1.2088	1.1702

Table 3.2: Variance of Kum–IW distribution for various choices of parameters

a	b	β				
		3	4	5	6	7
1	1	0.845	0.271	0.134	0.080	0.053
	2	0.101	0.051	0.031	0.021	0.015
	3	0.047	0.026	0.017	0.012	0.009
	4	0.030	0.018	0.012	0.008	0.006
	5	0.022	0.014	0.009	0.007	0.005
2	1	1.342	0.383	0.176	0.101	0.065
	2	0.160	0.072	0.041	0.026	0.018
	3	0.074	0.037	0.022	0.015	0.011
	4	0.048	0.025	0.015	0.010	0.008
	5	0.035	0.019	0.012	0.008	0.006
3	1	1.758	0.469	0.208	0.115	0.073
	2	0.210	0.089	0.048	0.030	0.021
	3	0.097	0.045	0.026	0.017	0.012
	4	0.063	0.031	0.018	0.012	0.009
	5	0.046	0.023	0.014	0.009	0.007
4	1	2.130	0.542	0.233	0.127	0.079
	2	0.255	0.102	0.054	0.033	0.022
	3	0.118	0.053	0.029	0.019	0.013
	4	0.076	0.035	0.020	0.013	0.009
	5	0.056	0.027	0.016	0.010	0.007
5	1	2.472	0.606	0.255	0.137	0.084
	2	0.295	0.114	0.059	0.036	0.024
	3	0.137	0.059	0.032	0.020	0.014
	4	0.088	0.040	0.022	0.014	0.010
	5	0.065	0.030	0.017	0.011	0.008

Table 3.3: Skewness and Kurtosis of Kum–IW distribution for various choices of parameters

a	b	β				
		3	4	5	6	7
Skewness	1	5.605	3.535	2.806	2.425	2.189
	2	1.776	1.502	1.342	1.237	1.163
	3	1.194	1.037	0.940	0.873	0.825
	4	0.927	0.809	0.734	0.682	0.643
	5	0.766	0.667	0.603	0.558	0.525
Kurtosis	1	48.092	24.678	17.534	14.166	12.229
	2	8.103	6.974	6.327	5.910	5.620
	3	5.349	4.919	4.653	4.473	4.344
	4	4.426	4.178	4.021	3.914	3.836
	5	3.976	3.808	3.700	3.626	3.572

Table 3.1 and 3.2 indicates that for fixed a and β the mean and variance of Kum–IW distribution will decrease with increase in b . Table 3.3 indicates that for fixed a and b , the Skewness and Kurtosis both decreases with increase in β .

The q th percentile of the distribution is obtained by solving $F(x) = q$ for the variable X . The q th percentile for Kum–IW distribution is obtained by using (2.5) in $F(x) = q$, which turned out to be:

$$x = \left[-\frac{a\alpha}{\ln\{1 - (1 - q)^{1/b}\}} \right]^{1/\beta} \tag{3.2}$$

The Median of the distribution is obtained by using $q=0.5$ in (3.2). The random sample can also be easily generated from (3.2) by using q as uniform random number.

The survivorship function for Kum–IW distribution is immediately written from (2.5) and is given as:

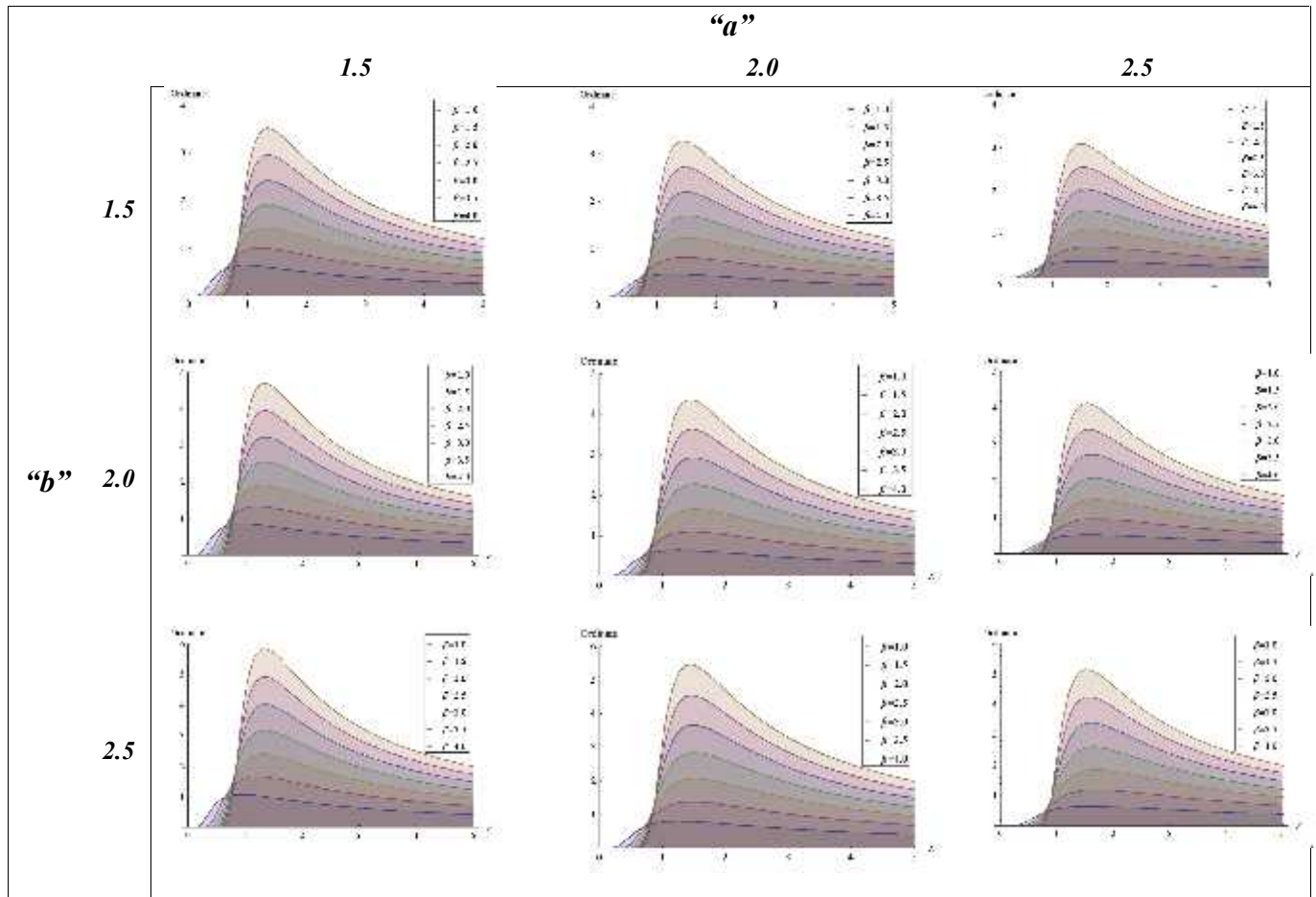
$$S_{K-IW}(t) = 1 - F_{K-IW}(t) = \left[1 - \exp\left(-\frac{a\alpha}{x^\beta}\right) \right]^b \tag{3.3}$$

The hazard rate function of the distribution is:

$$h_{K-IW}(t) = \frac{f_{K-IW}(t)}{1 - F_{K-IW}(t)} = \frac{ab\alpha\beta}{x^{\beta+1}} \frac{\exp\left(-\frac{a\alpha}{x^\beta}\right)}{\left[1 - \exp\left(-\frac{a\alpha}{x^\beta}\right)\right]} \quad (3.4)$$

The plots of (3.4), for different choices of the parameters, are given in figure 2. These plots shows that for fixed a and b , the hazard rate has higher peak for large values of β .

Figure 2: Hazard rate function of $KIW(a, b, \alpha, \beta)$



4. Parameter Estimation

In this section we have given the maximum likelihood estimates of parameter of Kum–IW distribution. The likelihood function for a random sample of size n from density (2.3) is:

$$LF = L(x_1, \dots, x_n) = (ab\alpha\beta)^n \prod_{i=1}^n x_i^{-(\beta+1)} \exp\left(-a\alpha \sum_{i=1}^n x_i^{-\beta}\right) \prod_{i=1}^n \left[1 - \exp\left(-a\alpha x_i^{-\beta}\right)\right]^{b-1}$$

The Log of likelihood function is:

$$\begin{aligned} \ln(LF) = n \ln(a) + n \ln(b) + n \ln(\alpha) + n \ln(\beta) - (\beta + 1) \sum_{i=1}^n \ln(x_i) - a\alpha \sum_{i=1}^n x_i^{-\beta} \\ + (b-1) \sum_{i=1}^n \ln[1 - \exp(-a\alpha x_i^{-\beta})] \end{aligned} \quad (4.1)$$

The derivatives of log of likelihood function *w.r.t.* the unknown parameters are:

$$\frac{\partial \ln(LF)}{\partial a} = \frac{n}{a} - \alpha \sum_{i=1}^n x_i^{-\beta} + (b-1) \sum_{i=1}^n \frac{\alpha x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})} \quad (4.2)$$

$$\frac{\partial \ln(LF)}{\partial b} = \frac{n}{b} \sum_{i=1}^n \ln[1 - \exp(-a\alpha x_i^{-\beta})] \quad (4.3)$$

$$\frac{\partial \ln(LF)}{\partial \alpha} = \frac{n}{\alpha} - a \sum_{i=1}^n x_i^{-\beta} + (b-1) \sum_{i=1}^n \frac{a x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})} \quad (4.4)$$

$$\frac{\partial \ln(LF)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \ln x_i + a\alpha \sum_{i=1}^n (\ln x_i) x_i^{-\beta} + (b-1) \sum_{i=1}^n \frac{a\alpha x_i^{-\beta} (\ln x_i) \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})} \quad (4.5)$$

The maximum likelihood estimates of a, b, α and β can be obtained by solving equations (4.2) to (4.5) simultaneously. The observed information matrix for the parameters of Kum–IW distribution is given as:

$$J(\boldsymbol{\theta}) = - \begin{bmatrix} J_{aa} & J_{ab} & J_{a\alpha} & J_{a\beta} \\ & J_{bb} & J_{b\alpha} & J_{b\beta} \\ & & J_{\alpha\alpha} & J_{\alpha\beta} \\ & & & J_{\beta\beta} \end{bmatrix}; \quad (4.6)$$

where $J_{\theta_j \theta_k} = \partial^2 \ln(LF) / \partial \theta_j \partial \theta_k$. These entries are given in the Appendix.

5. Application

In this section we have given the real data application of Kum–IW distribution. We have fitted the Kum – IW distribution for two different data sets. The results are given in Table 5.1 below. The results of this table clearly indicates that the proposed Kum–IW distribution fits well the given data sets in comparison with other distributions as this distribution has smallest AIC values.

Table 5.1: Parameter Estimates for Various Distributions

	Model	Parameters (SE in Parenthesis)				-2 Log (LF)	AIC
		a	b	Alpha	Beta		
Survival Time	K-IW	19.95(2.95)	3195.14(92.14)	0.45(0.07)	0.73(0.02)	-7614.66	-7606.66
	Inverse Weibull			11.81(0.84)	0.79(0.02)	11848.51	11852.51
	Weibull			52.38(0.99)	1.61(0.04)	11362.08	11366.08
	Gamma			2.10(0.08)	0.04(0.002)	11392.86	11396.86
Salary	K-IW	837.09(95.95)	6209.63(285.41)	881.68(100.72)	1.40(0.02)	1807.73	1815.73
	Inverse Weibull			485.72(83.30)	0.68(0.02)	10343.47	10347.47
	Weibull			584.47(59.32)	0.24(0.01)	11869.48	11873.48
	Gamma			2.10(0.08)	0.04(0.002)	11392.86	11396.86

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Appendix: Entries of Observed Information Matrix for Kum–IW Distribution

$$J_{aa} = \frac{\partial^2 \ln(LF)}{\partial^2 a} = -\frac{n}{a^2} - (b-1) \sum_{i=1}^n \frac{\alpha^2 x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2}$$

$$J_{ab} = \frac{\partial^2 \ln(LF)}{\partial a \partial b} = \sum_{i=1}^n \frac{\alpha x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{a\alpha} = \frac{\partial^2 \ln(LF)}{\partial a \partial \alpha} = -\sum_{i=1}^n x_i^{-\beta} - (b-1) \sum_{i=1}^n \frac{a\alpha x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2} + (b-1) \sum_{i=1}^n \frac{x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{a\beta} = \frac{\partial^2 \ln(LF)}{\partial a \partial \beta} = \alpha \sum_{i=1}^n (\ln x_i) x_i^{-\beta} + (b-1) \sum_{i=1}^n \frac{a\alpha^2 (\ln x_i) x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2} - (b-1) \sum_{i=1}^n \frac{\alpha (\ln x_i) x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{bb} = \frac{\partial^2 \ln(LF)}{\partial^2 b} = -\frac{n}{b^2}; \quad J_{b\alpha} = \frac{\partial^2 \ln(LF)}{\partial b \partial \alpha} = \sum_{i=1}^n \frac{a x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{b\beta} = \frac{\partial^2 \ln(LF)}{\partial b \partial \beta} = -\sum_{i=1}^n \frac{a\alpha (\ln x_i) x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{\alpha\alpha} = \frac{\partial^2 \ln(LF)}{\partial^2 \alpha} = -\frac{n}{\alpha^2} - (b-1) \sum_{i=1}^n \frac{a^2 x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2}$$

$$J_{\alpha\beta} = \frac{\partial^2 \ln(LF)}{\partial \alpha \partial \beta} = a \sum_{i=1}^n (\ln x_i) x_i^{-\beta} + (b-1) \sum_{i=1}^n \frac{a^2 \alpha (\ln x_i) x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2} + (b-1) \sum_{i=1}^n \frac{a (\ln x_i) x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$

$$J_{\beta\beta} = \frac{\partial^2 \ln(LF)}{\partial^2 \beta} = -\frac{n}{\beta^2} - a\alpha \sum_{i=1}^n (\ln x_i)^2 x_i^{-\beta} - (b-1) \sum_{i=1}^n \frac{a^2 \alpha^2 (\ln x_i)^2 x_i^{-2\beta} \exp(-a\alpha x_i^{-\beta})}{\{1 - \exp(-a\alpha x_i^{-\beta})\}^2} + (b-1) \sum_{i=1}^n \frac{a\alpha (\ln x_i)^2 x_i^{-\beta} \exp(-a\alpha x_i^{-\beta})}{1 - \exp(-a\alpha x_i^{-\beta})}$$