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On the Distribution of the Peña – Rodríguez Portmanteau Statistic

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Abstract

Peña and Rodríguez (2002) introduced a portmanteau test for time series which turns out to be more powerful than those proposed by Ljung and Box (1986) and Monti (1994), and approximated its distribution by means of a two-parameter gamma random variable. A polynomially adjusted beta approximation is proposed in this paper. This approximant is based on the moments of the statistic, which can be estimated by simulation or determined by symbolic computations or numerical integration. Various types of time series processes such as AR(1), MA(1), ARMA$(2,2)$ are being considered. The proposed approximation turns out to be nearly exact.

Keywords: Portmanteau test, Moments, Gamma approximation, Beta approximation, Symbolic computation.

1. Introduction

The zero-mean autoregressive moving average process of order (p,q) is defined to be a stationary and invertible solution of the equation, $\phi(B)X_t = \theta(B)\varepsilon_t$, where $\varepsilon_t \sim N(0,\sigma_\varepsilon^2)$, $\phi(B) = 1-\phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1-\theta_1 B - \cdots - \theta_q B^q$, the backshift operator B being such that $B^k X_t = X_{t-k}$. The X_t's often result from some transformation of an observed time series such as differencing. The residuals of this model are given by $\hat{\varepsilon}_t = \hat{\theta}^{-1}(B)\hat{\phi}(B)X_t$, where $\hat{\theta}(B)$ and $\hat{\phi}(B)$ are polynomials whose coefficients are taken to be the maximum likelihood estimates of the corresponding parameters. Several authors such as Box and Pierce (1970), Ljung and Box (1978), McLeod and Li (1983) and Monti

(1994) proposed diagnostic goodness of fit tests based on the lag k autocorrelation coefficients of the residuals, $\hat{\varepsilon}_t$, given by $r_k = \sum_{t=k+1}^{n} \hat{\varepsilon}_t \hat{\varepsilon}_{t-k} / \sum_{t=1}^{n} \hat{\varepsilon}_t^2$ for $k = 1, 2, \ldots$. Peña and Rodríguez (2002) suggested a more powerful portmanteau test whose asymptotic distribution is chi-square. Lin and McLeod (2006) pointed out that the convergence of this test statistic to its asymptotic distribution can be quite slow.

The Peña-Rodríguez portmanteau statistic is defined in Section 2. A symbolic computation methodology as well as a technique involving a recursive formula from which the moments of the statistic can be determined are described in Section 3. A polynomially adjusted beta density approximation is introduced in Section 4. In Section 5, such density approximations are shown to be more accurate than the approximations proposed in Peña and Rodríguez (2002) and (2006).

2. The Peña-Rodríguez Portmanteau Statistic

For stationary time series, the residual correlation matrix of order m is given by

$$\hat{R}_m = \begin{pmatrix} 1 & r_1 & \cdots & r_m \\ r_1 & 1 & \cdots & r_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_m & r_{m-1} & \cdots & 1 \end{pmatrix}. \quad (1)$$

Peña and Rodríguez (2002) proposed the following statistic to test for autocorrelations in the estimated residuals up to lag m:

$$\hat{D}_m = n[1 - |\hat{R}_m|^{1/m}], \quad (2)$$

and approximated its distribution by means of a gamma random variable with mean $(m+1)/2-(p+q)$ and variance $(m+1)(2m+1)/3m-2(p+q)$, assuming that the underlying process is ARMA(p,q). Peña and Rodríguez (2006) showed that a more accurate approximation can be obtained by making use of

$$D_m^{\star} = -\frac{n}{m+1} \log|\hat{R}_m|, \quad (3)$$

which is approximately distributed as a gamma random variable with mean $m/2-(p+q)$ and variance $[m(2m+1)/3(m+1)]-2(p+q)$. We are proposing a more accurate approximation that is based on the moments of $|\hat{R}_m|$.

3. Moments of the Determinant of the Sample Autocorrelation Matrix

Two techniques are being proposed for determining the moments of $|\hat{R}_m|$. First, we discuss the symbolic computational approach.

By making use of symbolic computational packages such as Maple or *Mathematica*, one can define an expected value operator E having the following properties:

$$E[\sum_{i=1}^{p} \alpha_i Y_i] \quad = \quad \sum_{i=1}^{p} \alpha_i \, E(Y_i)$$

and

$$E\left(\prod_{i=1}^{p} Y_i^{s_i}\right) = \prod_{i=1}^{p} E(Y_i^{s_i}),$$

where the α_i's and s_i's are constants and the Y_i's are independently distributed random variables, $i = 1,\ldots,p$. The *Mathematica* code needed to implement this operator is provided in Appendix 2.

In order to determine the moments of $|\hat{R}_m|$, we express the elements of the residual correlation matrix, $\hat{R}_m$, in terms of the quadratic forms, $Q_i = \varepsilon' A_i \varepsilon$ where $\varepsilon \sim N_n(\mathbf{0}, I)$ and $A_i = L_i + L_i'$, L_k being a null matrix with the zeros in its k^{th} subdiagonal replaced by $1/2$. Then, on expanding the determinant of $\hat{R}_m$, one has a sum of products of quadratic forms times $Q_0^{-(m+1)}$. The h^{th} moment of $|\hat{R}_m|$ is

$$E\left(|W_m|^h / Q_0^{h(m+1)}\right) \tag{4}$$

where

$$W_m = \begin{pmatrix} Q_0 & Q_1 & \cdots & Q_m \\ Q_1 & Q_0 & \cdots & Q_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_m & Q_{m-1} & \cdots & Q_0 \end{pmatrix}.$$

In light of Corollary 1 which is proved in Appendix 1, Q_0 and $Q_i / Q_0, i = 1,2,\ldots,m,$ are independently distributed, and therefore Q_0 and $|\hat{R}_m|$, which is a function of $Q_1/Q_0,\ldots,$ Q_m/Q_0, are also independently distributed. As a result,

$$E\left(|\hat{R}_m|^h\right) = E\left(|W_m|^h\right) / E(Q_0^{h(m+1)}). \tag{5}$$

Since, $Q_0 = \varepsilon' A_0 \varepsilon$ with $A_0 = I$, Q_0 is distributed as a chi-square random variable with n degrees of freedom and its $h(m+1)^{\text{th}}$ moment is

$$\frac{2^{h(m+1)}\Gamma(h(m+1) + n/2)}{\Gamma(n/2)}. \tag{6}$$

In order to obtain the h^{th} moment of $|W_m|$, we first expand the determinant. This yields a sum of the products of quadratic forms. Then expressing each of the quadratic forms as

$$Q_k = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)} \varepsilon_i \varepsilon_j, \; k = 0,1,\ldots,m, \tag{7}$$

where the $a_{ij}^{(k)}$'s are the elements of the matrix A_k, and expanding, one obtains a linear combination of products of powers of independent standard Gaussian random variables, which on application of the expected value operator yields the h^{th} moment of $|W_m|$.

For example, the second moment of $|\hat{R}_2|$ can be evaluated as follows when $n=3$. Letting $h=2$, Equation (4) becomes

$$\frac{\Gamma(\frac{3}{2})}{2^6\Gamma(\frac{3}{2}+6)} E(Q_0^3 - 2Q_0Q_1^2 + 2Q_1^2Q_2 - Q_0Q_2^2)^2$$

$$= \frac{\Gamma(\frac{3}{2})}{2^6\Gamma(\frac{3}{2}+6)} E(Q_0^6 - 4Q_0^4Q_1^2 + 4Q_0^2Q_1^4 + 4Q_0^3Q_1^2Q_2 - 8Q_0Q_1^4Q_2 - 2Q_0^4Q_2^2 \quad (8)$$

$$+4Q_0^2Q_1^2Q_2^2 + 4Q_1^4Q_2^2 - 4Q_0Q_1^2Q_2^3 + Q_0^2Q_2^4)$$

where $Q_k = \sum_{i=1}^{3-k}\varepsilon_i\varepsilon_{i+k}$, so that $Q_0^2 = \varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2$, $Q_1^2 = (\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3)^2$ and $Q_2^2 = (\varepsilon_1\varepsilon_3)^2$, which on expanding and simplifying gives

$$\frac{1}{135135} E(\varepsilon_1^{12} + 2\varepsilon_1^{10}\varepsilon_2^2 + 3\varepsilon_1^8\varepsilon_2^4 + 4\varepsilon_1^6\varepsilon_2^6 + 3\varepsilon_1^4\varepsilon_2^8 + 2\varepsilon_1^2\varepsilon_2^{10}$$
$$+\varepsilon_2^{12} - 4\varepsilon_1^9\varepsilon_2^2\varepsilon_3 - 12\varepsilon_1^7\varepsilon_2^4\varepsilon_3 - 12\varepsilon_1^5\varepsilon_2^6\varepsilon_3 - 12\varepsilon_1^3\varepsilon_2^8\varepsilon_3 - 8\varepsilon_1\varepsilon_2^{10}\varepsilon_3$$
$$+4\varepsilon_1^{10}\varepsilon_3^2 + 14\varepsilon_1^8\varepsilon_2^2\varepsilon_3^2 + 20\varepsilon_1^6\varepsilon_2^4\varepsilon_3^2 + 32\varepsilon_1^4\varepsilon_2^6\varepsilon_3^2 + 28\varepsilon_1^2\varepsilon_2^8\varepsilon_3^2 + 2\varepsilon_2^{10}\varepsilon_3^2$$
$$-12\varepsilon_1^7\varepsilon_2^2\varepsilon_3^3 - 40\varepsilon_1^5\varepsilon_2^4\varepsilon_3^3 - 48\varepsilon_1^3\varepsilon_2^6\varepsilon_3^3 - 12\varepsilon_1\varepsilon_2^8\varepsilon_3^3 + 8\varepsilon_1^8\varepsilon_3^4 + 26\varepsilon_1^6\varepsilon_2^2\varepsilon_3^4$$
$$+43\varepsilon_1^4\varepsilon_2^4\varepsilon_3^4 + 32\varepsilon_1^2\varepsilon_2^6\varepsilon_3^4 + 3\varepsilon_2^8\varepsilon_3^4 - 16\varepsilon_1^5\varepsilon_2^2\varepsilon_3^5 - 40\varepsilon_1^3\varepsilon_2^4\varepsilon_3^5 - 12\varepsilon_1\varepsilon_2^6\varepsilon_3^5$$
$$+10\varepsilon_1^6\varepsilon_3^6 + 26\varepsilon_1^4\varepsilon_2^2\varepsilon_3^6 + 20\varepsilon_1^2\varepsilon_2^4\varepsilon_3^6 + 4\varepsilon_2^6\varepsilon_3^6 - 12\varepsilon_1^3\varepsilon_2^2\varepsilon_3^7 - 12\varepsilon_1\varepsilon_2^4\varepsilon_3^7$$
$$+8\varepsilon_1^4\varepsilon_3^8 + 14\varepsilon_1^2\varepsilon_2^2\varepsilon_3^8 + 3\varepsilon_2^4\varepsilon_3^8 - 4\varepsilon_1\varepsilon_2^2\varepsilon_3^9 + 4\varepsilon_1^2\varepsilon_3^{10} + 2\varepsilon_2^2\varepsilon_3^{10} + \varepsilon_3^{12}),$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are independently distributed $N(0,1)$ random variables whose k^{th} moment is 0 when k is odd and $2^{k/2}\Gamma\left(\frac{k+1}{2}\right)/\sqrt{\pi}$ when k is even. On replacing $\varepsilon_i^k, i=1,2,3,$ by the corresponding moments, one has $E(|\hat{R}_2|^2) = 24412/45045$. Similarly, it can be verified that the first, third and fourth moments are respectively $74/105, 193184/440895$ and $6116550/16731965$.

The second approach is based on a general recursive formula for obtaining joint moments from joint cumulants. Letting $Q_i = \varepsilon' A_i \varepsilon$, $i=1,\ldots,\eta$, where A_i is a symmetric matrix and $\varepsilon \sim N_n(\mathbf{0},V)$, the joint cumulant generating function of $Q_1,\ldots,Q_\eta$ is

$$K_{Q_1,\ldots,Q_\eta}(t_1,\ldots,t_\eta) = \ln|I-W|^{-\frac{1}{2}} = \frac{1}{2}\sum_{j=1}^{\infty}\text{tr}(W^j)/j \quad (9)$$

where $W = 2\sum_{i=1}^{\eta}(t_iVA_i)$, see for instance Mathai and Provost (1992, Section 3.3). The joint moments, $E[(Q_1-E(Q_1))^{s_1}\cdots(Q_\eta-E(Q_\eta))^{s_\eta}] \equiv \mu_{s_1,\ldots,s_\eta}$, (in this case, $E(Q_i)=\text{tr}(A_iV)$ $i=1,\ldots,\eta$) can then be determined from the joint cumulants by making use of the following recursive relationship derived for instance by Smith (1995) :

$$\mu_{s_1,s_2,\ldots,s_{m+1}} = \sum_{i_1=0}^{s_1}\cdots\sum_{i_m=0}^{s_m}\sum_{i_{m+1}=0}^{s_{m+1}-1}\binom{s_1}{i_1}\cdots\binom{s_m}{i_m}\binom{s_{m+1}-1}{i_{m+1}} \times K_{s_1-i_1,s_2-i_2,\ldots,s_{m+1}-i_{m+1}}\ \mu_{i_1,i_2,\ldots,i_{m+1}},\ m=1,2,\ldots,\eta-1, \qquad (10)$$

where $\mu_{0,0,\ldots,0}=1$ and $K_{a_1,\ldots,a_{m+1}}$ denotes the joint cumulant of $Q_1,\ldots,Q_{m+1}$ of orders $a_1,\ldots,a_{m+1}$, which is equal to

$$\frac{\partial^{a_1+\cdots+a_{m+1}}}{\partial t_1^{a_1}\cdots\partial t_{m+1}^{a_{m+1}}}\ K_{Q_1,\ldots,Q_{m+1}}(t_1,\ldots,t_{m+1}) \text{ evaluated at } t_i=0,\ i=1,\ldots,m+1.$$

For example, on applying Corollary 1 of Appendix 1, the h^{th} moment of $|\hat{R}_2|$ is determined by evaluating

$$\Gamma(n/2)/2^{h(m+1)}\Gamma(h(m+1)+n/2)\ E(Q_0^3-2Q_0Q_1^2+2Q_1^2Q_2-Q_0Q_2^2)^h.$$

In particular, in order to obtain the second moment when $n=3$, one has

$$\begin{aligned} E\left(|\hat{R}_2|^2\right) &= (1/135135)\ E(Q_0^6-4Q_0^4Q_1^2+4Q_0^2Q_1^4+4Q_0^3Q_1^2Q_2-8Q_0Q_1^4Q_2-2Q_0^4Q_2^2 \\ &\quad +4Q_0^2Q_1^2Q_2^2+4Q_1^4Q_2^2-4Q_0Q_1^2Q_2^3+Q_0^2Q_2^4) \\ &= (1/135135)\{E(Q_0^6)-4E(Q_0^4Q_1^2)+4E(Q_0^2Q_1^4)+4E(Q_0^3Q_1^2Q_2) \\ &\quad -8E(Q_0Q_1^4Q_2)-4E(Q_1^4Q_2^2)2E(Q_0^4Q_2^2)+4E(Q_0^2Q_1^2Q_2^2)+ \\ &\quad -4E(Q_0Q_1^2Q_2^3)+E(Q_0^2Q_2^4)\} \end{aligned} \qquad (11)$$

where for instance

$$E(Q_0^6) = \mu_{6,0,0}=135135$$

is obtained from

$$\mu_{\ell,0,0} = \sum_{i=0}^{\ell}\binom{l}{i}K_{\ell-i,0,0}\ \mu_{i,0,0}, \qquad \ell=1,2,\ldots,6,$$

with $\mu_{0,0,0}=1$ and

$$\begin{aligned} K_{\ell,0,0} &= \frac{\partial^\ell}{\partial t_1^\ell}\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{\text{tr}(2t_1A_1)^j}{j}\right)\text{evaluated at } t_1=0 \\ &= 2^{\ell-1}(\ell-1)!\,\text{tr}A_1^\ell\ ; \\ 4E(Q_0^4\,Q_1^2) &= 4\mu_{4,2,0} \\ &= 4\sum_{i=0}^{4}\sum_{j=0}^{1}\binom{4}{i}\binom{1}{j}K_{4-i,2-j,0}\ \mu_{i,j,0}=72072 \end{aligned}$$

with

$$\mu_{r,t,0} = \sum_{i=0}^{r}\sum_{j=0}^{t-1}\binom{r}{i}\binom{t-1}{j}K_{r-i,t-j,0}\;\mu_{i,j,0}, \quad r=0,1,\dots,4,$$

$$K_{h,\ell,0} = \frac{\partial^{h+\ell}}{\partial t_1^h \partial t_2^\ell}K_{Q_1,Q_2}(t_1,t_2) \text{ evaluated at } t_1=0,\ t_2=0$$

$$= \frac{h!\,\ell!\,2^{h+\ell}}{2(h+\ell)}\operatorname{tr}\sum_{(h,\ell)}(A_1A_2),$$

the notation $\sum_{(h,\ell)}(A_1A_2)$ standing for the sum of all the possible distinct permutations of a product of h matrices A_1 and ℓ matrices A_2; and

$$4E(Q_0^2\,Q_1^2\,Q_2^2) = 4\,\mu_{2,2,2}$$

$$= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{1}\binom{2}{i}\binom{2}{j}\binom{1}{k}K_{2-i,2-j,2-k}\;\mu_{i,j,k} = 3432$$

with

$$\mu_{r,t,s} = \sum_{i=0}^{r}\sum_{j=0}^{t}\sum_{k=0}^{s-1}\binom{r}{i}\binom{t}{j}\binom{s-1}{j}K_{r-i,t-j,s-k}\;\mu_{i,j,k}, \quad r=0,1,2,\ t=0,1,2,$$

$$s=1,$$

$$K_{h,\ell,q} = \frac{\partial^{h+\ell+q}}{\partial t_1^h \partial t_2^\ell \partial t_3^q}K_{Q_1,Q_2,Q_3}(t_1,t_2,t_3) \text{ evaluated at } t_1=0,\ t_2=0,\ t_3=0$$

$$= \frac{h!\,\ell!\,q!2^{h+\ell+q}}{2(h+\ell+q)}\operatorname{tr}\sum_{(h,\ell,q)}(A_1A_2A_3),$$

the notation $\sum_{(h,\ell,q)}(A_1A_2A_3)$ standing for the sum of all the possible distinct permutations of a product of h matrices A_1, ℓ matrices A_2 and q matrices of A_3. The other terms in Equation (10) can be evaluated similarly. They are

$$4\,\mu_{2,4,0} = 4\sum_{i=0}^{2}\sum_{j=0}^{3}\binom{2}{i}\binom{3}{j}K_{2-i,4-j,0}\;\mu_{i,j,0} = 20592,$$

$$4\,\mu_{3,2,1} = 4\sum_{i=0}^{3}\sum_{j=0}^{2}\binom{3}{i}\binom{2}{j}K_{3-i,2-j,1-k}\;\mu_{i,j,k} = 10296,$$

$$8\,\mu_{1,2,1} = 8\sum_{i=0}^{1}\sum_{j=0}^{2}\binom{1}{i}\binom{2}{j}K_{1-i,2-j,1-k}\;\mu_{i,j,k} = 7488,$$

$$2\,\mu_{4,0,2} = 2\sum_{i=0}^{4}\sum_{k=0}^{1}\binom{4}{i}\binom{1}{k}K_{4-i,0,2-k}\;\mu_{i,0,k} = 18018,$$

$$4\,\mu_{0,4,2} = 4\sum_{j=0}^{4}\sum_{k=0}^{1}\binom{4}{j}\binom{1}{k}K_{0,4-j,2-k}\;\mu_{0,j,k} = 1008,$$

$$4\,\mu_{1,2,3} = 4\sum_{i=0}^{1}\sum_{j=0}^{2}\sum_{k=0}^{2}\binom{1}{i}\binom{2}{j}\binom{2}{k}K_{1-i,2-j,3-k}\;\mu_{i,j,k} = 936,$$

and

$$\mu_{2,0,4} = \sum_{i=0}^{2}\sum_{k=0}^{3}\binom{2}{i}\binom{3}{k}K_{2-i,0,4-k}\ \mu_{i,0,k} = 1287.$$

Thus the second moment of $|\hat{R}_2|$ when $n=3$ is equal to $24412/45045$. The moments so obtained are of course identical to those determined by means of the symbolic computational approach. We found the recursive formula to be computationally more efficient. When neither of these approaches is applicable, the moments can be determined by simulation or numerical integration.

4. Polynomially-Adjusted Beta Density Approximants

It is shown in this section that given the moments of $|\hat{R}_m|$, its distribution can be approximated in terms of an initial beta distributed approximant.

Let Y be a random variable defined in the closed interval $[a,b]$, whose raw moments $E(Y^h)$ are denoted by $\mu_Y(h)$, $h=0,1,\ldots$. First, the support of Y is mapped onto the interval [0,1] by means of the transformation

$$X = (Y-a)/(b-a). \tag{12}$$

Accordingly, the j^{th} moment of X is

$$\mu_X(j) = \sum_{h=0}^{j}\frac{\mu_Y(h)}{(b-a)^j}(-a)^{j-h}. \tag{13}$$

Then, on the basis of the first d moments of X, a density approximation of the following form is assumed for X:

$$g(x) = \psi(x)\sum_{j=0}^{d}\xi_j x^j, \tag{14}$$

where $\psi(x)$ is a base density function and $\sum_{j=0}^{d}\xi_j x^j$ is a polynomial adjustment. In this case, the base density is assumed to be that of a $beta(\alpha+1,\beta+1)$ random variable. Thus

$$\psi(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}x^{\alpha}(1-x)^{\beta}, \quad 0<x<1, \tag{15}$$

where

$$\alpha = \mu_X(1)\frac{\mu_X(1)-\mu_X(2)}{\mu_X(2)-\mu_X(1)^2}-1 \tag{16}$$

and

$$\beta = (1-\mu_X(1))\frac{\alpha+1}{\mu_X(1)}-1. \tag{17}$$

Its j^{th} moment is given by

$$m(j) = \frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+1+j)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2+j)} = \frac{\prod_{k=0}^{j}(\alpha+1+k)}{\prod_{k=0}^{j}(\alpha+\beta+2+k)}.$$

The coefficients ξ_j are determined by equating the h^{th} moment of X to the h^{th} moment obtained from the approximate distribution specified by $g(x)$. That is,

$$\begin{aligned}\mu_X(h) &= \int_0^1 x^h \psi(x) \sum_{j=0}^{d} \xi_j x^j \mathrm{d}x \\ &= \sum_{j=0}^{d} \xi_j \int_0^1 x^{h+j} \psi(x) \mathrm{d}x \\ &= \sum_{j=0}^{d} \xi_j m(h+j), \quad h = 0,1,\ldots,d,\end{aligned}$$

which yields a system of linear equations whose solution is

$$\begin{pmatrix}\xi_0\\ \xi_1\\ \vdots\\ \xi_d\end{pmatrix} = \begin{pmatrix} m(0) & m(1) & \cdots & m(d-1) & m(d)\\ m(1) & m(2) & \cdots & m(d) & m(d+1)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ m(d) & m(d+1) & \cdots & m(2d-1) & m(2d)\end{pmatrix}^{-1} \begin{pmatrix}1\\ \mu_X(1)\\ \vdots\\ \mu_X(d)\end{pmatrix}. \quad (18)$$

Finally, in light of the transformation specified by (12), the beta polynomial density approximant of Y is given by

$$\frac{1}{(b-a)} g\left(\frac{y-a}{b-a}\right) \quad (19)$$

for $a < y < b$.

5. Simulation Studies

Five types of processes are being considered:

Gaussian	$X_t = \varepsilon_t$;
AR(1)	$X_t = \varepsilon_t + 0.5Y_{t-1}$;
MA(1)	$X_t = \varepsilon_t + 0.5\varepsilon_{t-1}$;
ARMA(1,1)	$X_t = \varepsilon_t + 0.7Y_{t-1} + 0.4\varepsilon_{t-1}$;
ARMA(2,2)	$X_t = \varepsilon_t + 0.9Y_{t-1} - 0.4Y_{t-1} + 1.2\varepsilon_{t-1} - 0.3\varepsilon_{t-2}$;

where the ε_t's are independently distributed standard normal variables.

Note that for non-Gaussian processes with associated covariance matrix V, the quadratic forms Q_i defined in Section 3 are in fact equal to $(V^{-1/2}\,\varepsilon\,)' A_i (V^{-1/2}\,\varepsilon\,)$, where V is the covariance matrix associated with a given process. The covariance matrices associated with $\text{MA}(1)$ and $\text{AR}(1)$ processes can be obtained for instance from Box and Jenkins (1976) p.57 and p.69, respectively, while those associated with $\text{ARMA}(1,1)$ and ARMA $(2,2)$ processes are available for example from the *Mathematica* package *InverseCovarianceMatrixARMA* prepared by McLeod (2005).

After obtaining the moments of $|\hat{R}_m|$ either from the techniques described in Section 3 or by simulations (or numerical integration), we determined a polynomially-adjusted density approximant as defined in Section 4—with $a=0$ and $b=1$.

In order to compare our approximation to the distribution of $|\hat{R}_m|$ with Peña and Rodríguez's two-parameter gamma approximations to the density functions of $\hat{D}_m$ as defined in Equation (2), we apply a certain change of variables to the latter ones.

Peña and Rodríguez's (2002) proposed approximating the density function of $\hat{D}_m$ with

$$f_{\hat{D}_m}(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \qquad x>0,$$

where α and β are specified by (22) and (23). Since, according to Equation (2),

$$|\hat{R}_m| = (1-\frac{\hat{D}_m}{n})^m, \tag{20}$$

one has the following density approximation for $|\hat{R}_m|$:

$$g_{|\hat{R}_m|}(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}[n(1-y^{\frac{1}{m}})]^{(\alpha-1)} e^{-\beta[n(1-y^{\frac{1}{m}})]}\,|\,n(-\frac{1}{m}y^{\frac{1}{m}-1})\,|, \tag{21}$$

where

$$\alpha = \frac{3m[(m+1)-2(p+q)]^2}{2[2(m+1)(2m+1)-12m(p+q)]} \tag{22}$$

and

$$\beta = \frac{3m[(m+1)-2(p+q)]}{2(m+1)(2m+1)-12m(p+q)}, \tag{23}$$

which will be referred to as the first transformed gamma density.

Similarly, one has from the approximation proposed in Peña and Rodríguez's (2002) as given in Equation (3)

$$|\hat{R}_m| = \exp\{-\frac{m+1}{n}D_m^{\star}\}. \tag{24}$$

Since $D_m^{\star}$ follows a gamma distribution, one has

$$h_{|\hat{R}_m|}(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}[-\frac{n}{m+1}\log(y)]^{(\alpha-1)} e^{-\beta[-\frac{n}{m+1}\log(y)]}\,|-\frac{n}{m+1}\frac{1}{y}|, \tag{25}$$

which will be referred to as the second transformed gamma density, where

$$\alpha = \frac{3(m+1)\{m-2(p+q)\}^2}{2\{2m(2m+1)-12(m+1)(p+q)\}} \tag{26}$$

and

$$\beta = \frac{3m(m+1)\{m-2(p+q)\}}{2m(2m+1)-12(m+1)(p+q)}. \tag{27}$$

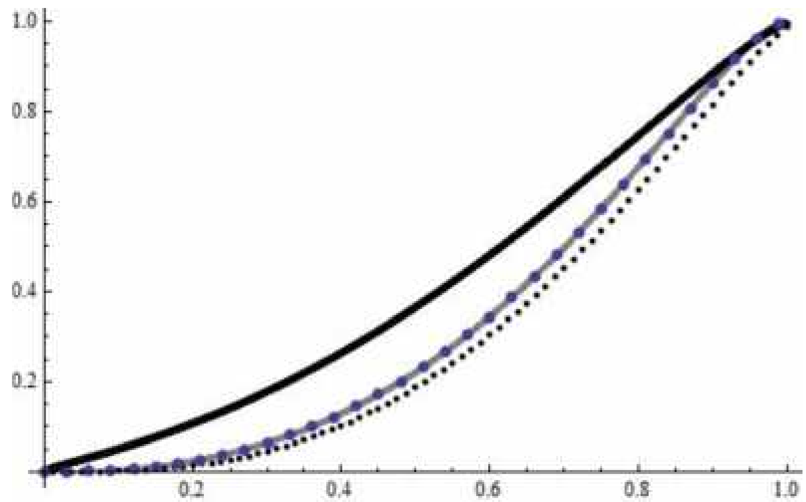

Figure 1: First transformed gamma CDF approximation (black line), second transformed gamma CDF approximation (dots) and proposed polynomially adjusted beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 10$ and $m = 3$ [Normal process]

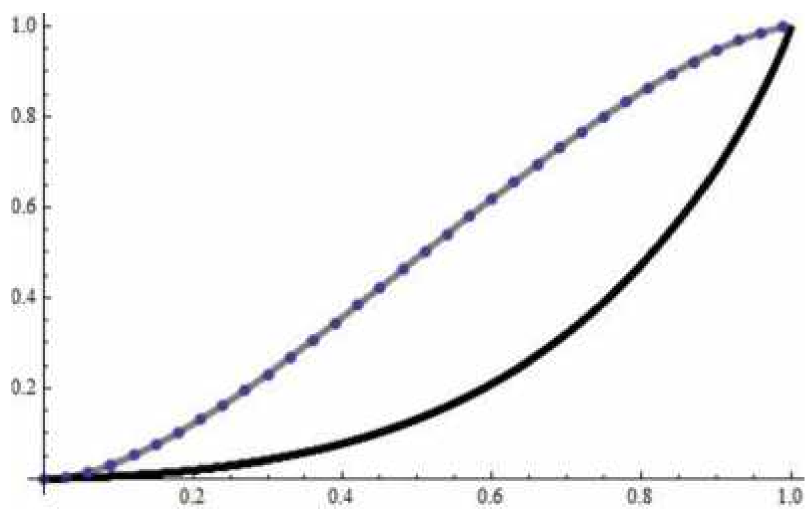

Figure 2: First transformed gamma CDF approximation (black line) and proposed polynomially adjusted beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 10$ and $m = 3$ [$\text{MA}(1)$ process]

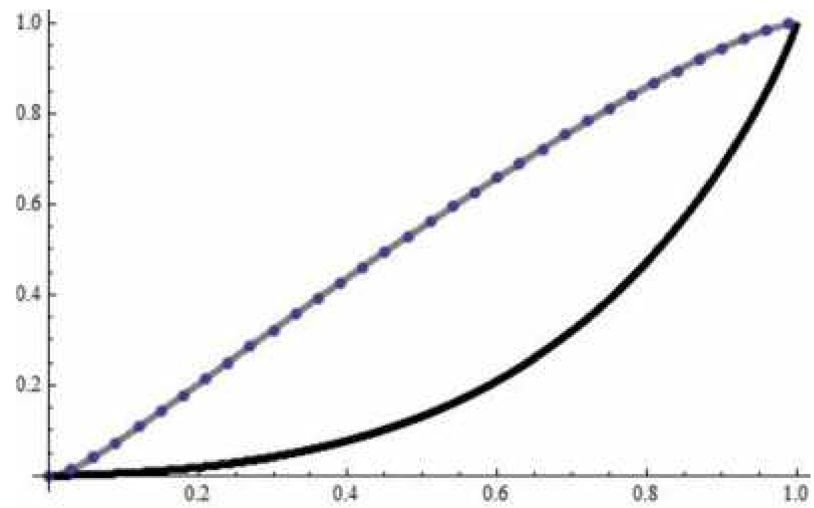

Figure 3: First transformed gamma CDF approximation (black line) and proposed polynomially adjusted beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 10$ and $m = 3$ [$\text{AR}(1)$ process]

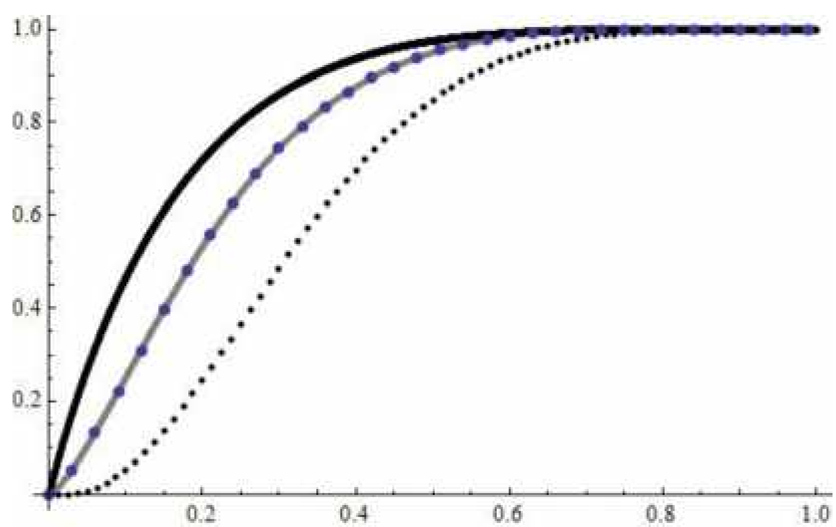

Figure 4: First transformed gamma CDF approximation (black line), second transformed gamma CDF approximation (dots) and Proposed Polynomially Adjusted Beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 36$ and $m = 12$ [Normal process]

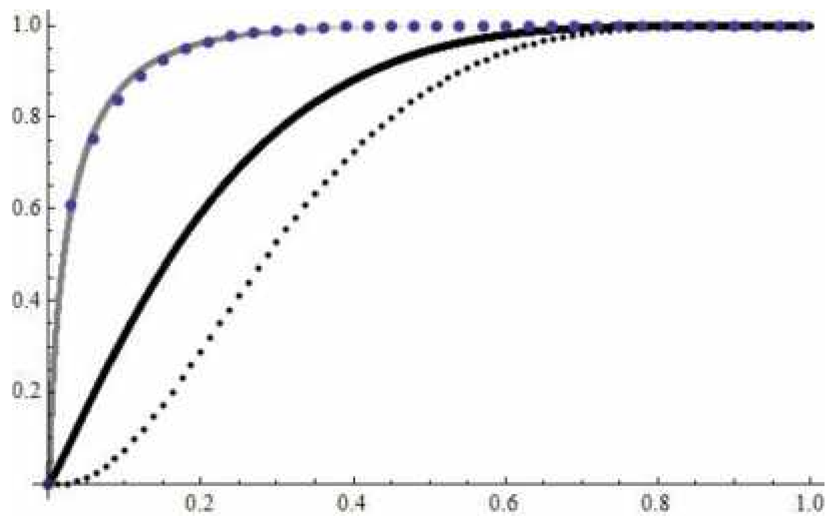

Figure 5: First transformed gamma CDF approximation (black line), second transformed gamma CDF approximation (dots) and beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 36$ and $m = 12$ [$\text{MA}(1)$ process]

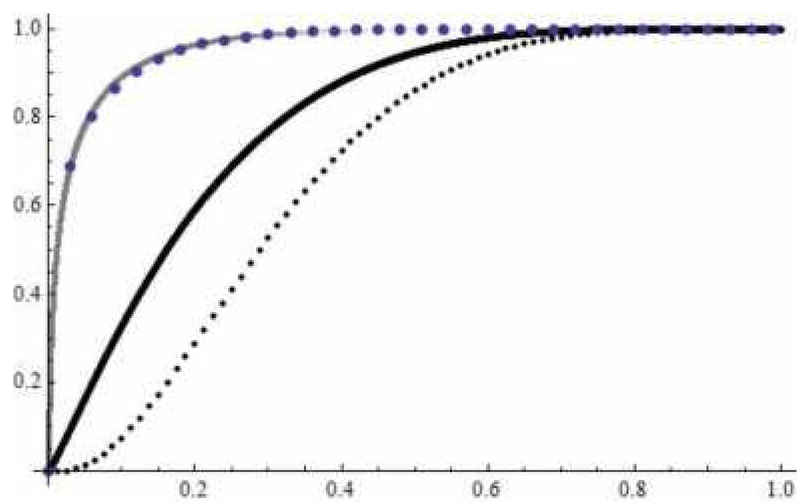

Figure 6: First transformed gamma CDF approximation (black line), second transformed gamma CDF approximation (dots) and beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 36$ and $m = 12$ [$\text{AR}(1)$ process]

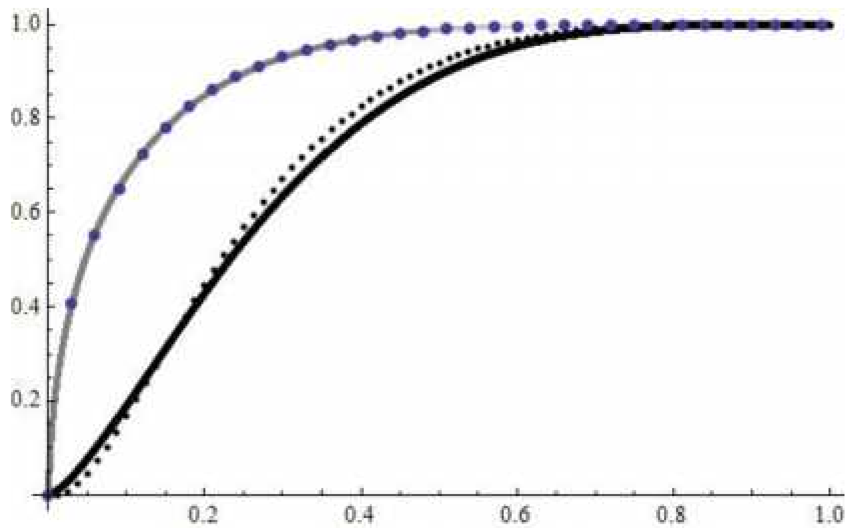

Figure 7: First transformed gamma CDF approximation (black line), second transformed gamma CDF approximation (dots) and beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 36$ and $m = 12$ [ARMA(1,1) process]

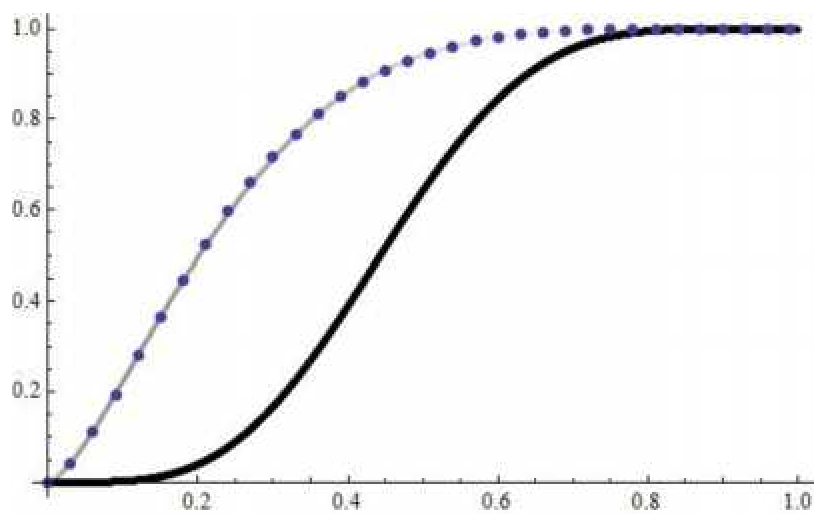

Figure 8: First transformed gamma CDF approximation (black line) and beta CDF approximation (large dots) superimposed on the simulated CDF (in grey) for $n = 36$ and $m = 12$ [ARMA(2,2) process]

Figures 1 through 8 include plots of the proposed polynomially adjusted beta cdf approximation (with $d = 8$) superimposed on the simulated cdf based on 100,000 replications and the Peña-Rodríguez transformed gamma approximations $g_{|\hat{R}_m|}(\cdot)$ and $h_{|\hat{R}_m|}(\cdot)$ for various types of processes with $n = 10$, $m = 3$ and $n = 36$, $m = 12$. Note that in light of (22) and (23), one needs $(p+q) < (m+1)(2m+1)/(6m)$. Accordingly, $g_{|\hat{R}_m|}(\cdot)$ is not defined for ARMA$(1,1)$ and ARMA$(2,2)$ processes when $m=3$. The second transformed gamma approximation which follows from that proposed in Peña and Rodríguez (2006) requires that $(p+q) < m(2m+1)/(6(m+1))$, so that this approximation is unavailable for the ARMA$(2,2)$ process when $m=12$.

Tables 1, 2 and 3 indicate that the polynomial adjustment yields even more accurate results, although the two-moment beta approximation already proves quite adequate.

CDF	Simulation	Beta Approximation	Adjusted Beta
0.01	0.13566	0.14968	0.13626
0.05	0.26905	0.27704	0.26962
0.10	0.36126	0.36273	0.36076
0.25	0.52649	0.52235	0.52686
0.50	0.70134	0.69799	0.70125
0.75	0.83881	0.84012	0.83869
0.90	0.92009	0.92476	0.91986
0.95	0.95144	0.95637	0.95117
0.99	0.98413	0.98729	0.98440

Table 1: Percentiles under a normal process (n = 10 and m = 3)

CDF	Simulation	Beta Approximation	Adjusted Beta
0.01	0.04764	0.04168	0.04850
0.05	0.11851	0.11288	0.11838
0.10	0.17862	0.17473	0.17815
0.25	0.31556	0.31680	0.31581
0.50	0.50941	0.51280	0.50860
0.75	0.70621	0.70601	0.70635
0.90	0.84562	0.84209	0.84544
0.95	0.90447	0.89989	0.90459
0.99	0.96862	0.96456	0.96805

Table 2: Percentiles under an MA(1) process (n = 10 and m = 3)

CDF	Simulation	Beta Approximation	Adjusted Beta
0.01	0.02417	0.01537	0.02420
0.05	0.06924	0.06045	0.06979
0.10	0.11402	0.10958	0.11397
0.25	0.23988	0.24373	0.23998
0.50	0.45444	0.45798	0.45395
0.75	0.68768	0.68430	0.68745
0.90	0.84452	0.84258	0.84485
0.95	0.90732	0.90648	0.90664
0.99	0.96887	0.97188	0.96934

Table 3: Percentiles under an AR(1) process (n = 10 and m = 3)

Acknowledgement

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Appendix 1. The Moments of Certain Ratios of Quadratic Forms

It is shown that R, the ratio of quadratic forms as defined in *Theorem 1*, is distributed independently of its denominator.

Theorem 1. Let $R = \mathbf{X}'PAP\mathbf{X} / \mathbf{X}'P\mathbf{X}$ where P is an idempotent matrix of rank $r \le p$, A is a symmetric matrix and $\mathbf{X} \sim N_p(\mathbf{0}, I)$; then R is distributed independently of $\mathbf{X}'P\mathbf{X}$.

Proof: Since P is idempotent, the matrices PAP and P commute, they can be diagonalized by means of the same orthogonal matrix. Let T be such an orthogonal matrix. Then

$$P = T\Lambda T' \tag{28}$$

$$= \sum_{i=1}^{p} \lambda_i \mathbf{t}_i \mathbf{t}'_i \,, \tag{29}$$

where the columns of T are the normalized eigenvectors corresponding to the eigenvalues of P and $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1, \ldots, \lambda_p$ being the eigenvalues of P. We note that in light of the representation of P given in (28), the λ_i's can be reordered (along with the $\mathbf{t}_i$'s). Since P is an idempotent matrix of rank r, r of the λ_i's will be equal to one while $p - r$ will be equal to zero. Thus we can take Λ to be Diag $(1, \ldots, 1, 0, \ldots, 0)$ where the first r elements of the diagonal are equal to one.

Now, consider the representation of PAP obtained from the spectral decomposition theorem, that is,

$$PAP = TDT'. \tag{30}$$

Since $P = T\Lambda T'$, we also have

$$PAP = T\Lambda T'AT\Lambda T'. \tag{31}$$

Then clearly, the diagonal matrix $D = \Lambda T'AT\Lambda$ will be of the form Diag $(\delta_1, \ldots, \delta_r, 0, \ldots, 0)$. Since T is an orthogonal matrix, $\mathbf{Z} = T'\,\mathbf{X} \sim N_p(\mathbf{0}, I)$, and we have

$$R = \frac{\mathbf{X}'PAP\mathbf{X}}{\mathbf{X}'P\mathbf{X}} = \frac{\mathbf{X}'TDT'\mathbf{X}}{\mathbf{X}'T\Lambda T'\mathbf{X}}$$

$$\sim \frac{\mathbf{Z}'D\mathbf{Z}}{\mathbf{Z}'\Lambda\mathbf{Z}} = \frac{\mathbf{Z}^{*'}\text{Diag}(\delta_1, \ldots, \delta_r)\mathbf{Z}^*}{\mathbf{Z}^{*'}\,\mathbf{Z}^*} = \frac{\sum_{k=1}^{r} \delta_k Z_k^2}{\sum_{k=1}^{r} Z_k^2}, \tag{32}$$

with $\mathbf{Z}^* = (Z_1, \ldots, Z_r)' \sim N_r(\mathbf{0}, I)$.

From a representation of spherically symmetric random vectors given by Cambanis, Huang and Simons (1981), we can express $\mathbf{Z}^*$, which has a spherically symmetric distribution, as the product of $\|\mathbf{Z}^*\|$ and $\mathbf{U}^{(r)}$, a random vector that is uniformly distributed on the unit hypersphere in $\Re^r$, $\|\mathbf{Z}^*\|$ and $\mathbf{U}^{(r)}$ being independently distributed. We note that $\mathbf{U}^{(r)}$ can be expressed as $\dfrac{\mathbf{Z}^*}{\|\mathbf{Z}^*\|}$ since

$$\mathbf{Z}^* = \|\mathbf{Z}^*\| \frac{\mathbf{Z}^*}{\|\mathbf{Z}^*\|}. \tag{33}$$

Thus, $\mathbf{Z}^{*\prime}\mathbf{Z}^* = \|\mathbf{Z}^*\|^2$ where $\mathbf{Z}^{*\prime}\mathbf{Z}^* \sim \mathbf{X}'P\mathbf{X}$ and $R \sim \sum_{k=1}^{r} \delta_k Z_k^2$, which is a function of the squares of the components of $\mathbf{Z}^*/\|\mathbf{Z}^*\|$, are independently distributed.

Corollary 1. Under the assumptions of *Theorem 1,*

$$E\left[\left(\frac{\mathbf{X}'PAP\mathbf{X}}{\mathbf{X}'P\mathbf{X}}\mathbf{X}'P\mathbf{X}\right)^h\right] = E\left[\left(\frac{\mathbf{X}'PAP\mathbf{X}}{\mathbf{X}'P\mathbf{X}}\right)^h\right] E\left[\left(\mathbf{X}'P\mathbf{X}\right)^h\right] \tag{34}$$

and thus,

$$E\left[\left(\frac{\mathbf{X}'PAP\mathbf{X}}{\mathbf{X}'P\mathbf{X}}\right)^h\right] = \frac{E[(\mathbf{X}'PAP\mathbf{X})^h]}{E[(\mathbf{X}'P\mathbf{X})^h]}. \tag{35}$$

Appendix 2. Mathematica Code to Obtain the Moments of $|W_m|$ Symbolically

```
ClearAll[A, K, m];
A[w_,n_]:=A[w,n] =
If[w > 0, Table[If[Abs[i − j]==w&&w > 0,1/2,0],{i,n},{j,n}], IdentityMatrix[n]];
RVQ[X_j_Integer] = True;

ClearAll[ε];
RVQ[g_[a__]]:=RVQ[a]; RVQ[a_, b__]:=If[RVQ[a], True, RVQ[b]];
RVQ[z_]:=False;
SetAttributes[ε,{Listable}]; ε[c_]:=c/;!RVQ[c];
HoldPattern[ε[Plus[a__]]]:=Map[ε, Plus[a]];
ε[c_ v_]:=c ε[v]/;!RVQ[c];
HoldPattern[ε[Times[a__]]]:=Map[ε, Times[a]];
ε[RV_^k_.] :=(2^k / Pi)^(1/2)Gamma[(k+1)/2](2IntegerPart[k/2]−k+1);
Q1[s_, n_, k_]:=Q1[s, n, k] = (Sum_{j=1}^n Sum_{i=1}^n A[k,n][[i, j]] X_i X_j)^(2s);

        << DiscreteMath Combinatorica
        c1[r_, m_]:=Compositions[r, m]
        Q1[(c1[2,2][[3,1]]), 3, 3]
        (X1 X2 + X2 X3)^4
        Expand[Product[Q1[(c1[2,2][[3,i]]),3,i] / (c1[2,2][[3,i]])!,{i,2,2}]]
        1/2 X1^4 X2^4 + 2 X1^3 X2^4 X3 + 3 X1^2 X2^4 X3^2 + 2 X1 X2^4 X3^3 + 1/2 X2^4 X3^4
        ClearAll[S6e];
        S6e[r_, n_, m_]:=
S6e[r, n, m] = (r!Gamma[n/2] / (2^2r Gamma[2r + n/2]))
Sum[e[Expand[Product[Q1[(c1[r,m][[j,i]]), n, i] / (c1[r,m][[j,i]])!,
{i,1,m}]]],{j,1,Length[c1[r,m]]}]
mf6 = Table[S6e[r,10,3],{r,0,4}];
mf6
{1, 1/5, 149/2240, 6529/201600, 762493/35481600}
mom = Table[mf6[[j]]10^(j-1),{j,1,5}]
{1, 2, 745/112, 32645/1008, 19062325/88704}
```

The *Mathematica* code for the other calculations is available from the authors upon request.