Application of A Second Derivative Multi-Step Method to Numerical Solution of Volterra Integral Equation of Second Kind

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Abstract

As is known, many problems of natural science are reduced mainly to the solution of nonlinear Volterra integral equations. The method of quadratures that was first applied by Volterra to solving variable boundary integral equations is popular among numerical methods for the solution of such equations. At present, there are different modifications of the method of quadratures that have bounded accuracies. Here we suggest a second derivative multistep method for constructing more exact methods.

Keywords: Volterra integral equations, Second derivative multistep method, Finite-difference method, Stability of finite- difference methods, Degree of finite-difference method.

Introduction

As is known, investigations of variable boundary integral equations began with Abel's known paper published in 1826. Volterra is a founder of the theory of variable boundary integral equations. He was the first who saw the importance of this theory and considered it systematically. Therefore, these equations are related with his name. Many famous scientists were engaged in approximate solution of Volterra integral equations. They have published several papers. A part of them were devoted to numerical solution of Volterra integral equations related with application of computer. The method of quadrature is more popular among the numerical methods. In this method, the volume of computations works increases at each integration step while passing from the current point to the next one. For removing the indicated deficiency the specialists suggested the Runge-Kutta, Adams and etc. methods together with the quadrature method. Here we suggest a method that allows to preserve the constant volume of calculations at each step.

Consider the following nonlinear Volterra integral equation that sometimes is called the Volterra-Uryson equation

$$y(x) = f(x) + \int_{x_0}^{x} K(x,s,y(s))ds, \quad x \in [x_0, X]. \qquad (1)$$

Assume that problem (1) has a unique continuous solution $y(x)$ determined on the interval $[x_0, X]$. By means of a constant step $0 < h$ divide the interval $[x_0, X]$ into N equal parts with the points $x_i = x_0 + ih$ (i = 0,1,...,N) and denote by y_i the approximate, and by $y(x_i)$ *exact value of the solution of problem (1) at the points* x_i *(i=0,1,2,…,N).*

For finding numerical solution of linear equation of type (1), Volterra applied the method of quadratures. Further, this method was revised and modified by many scientists (Corduneanu, 1991 and Manjirov and Polyanin, 2000). However, in these papers, the basic deficiency of the method of quadratures was not eliminated. This deficiency is that the volume of calculations increases while passing from one point to another one. Indeed, after applying the method of quadrature to equation (1), we have:

$$y_n = f_n + h\sum_{i=0}^{n} a_i K(x_n, x_i, y_i). \qquad (2)$$

This method was written for approximate value of the solution of equation (1), i.e. the quantity y_n at the point x_n. Obviously, while passing to the next point x_{n+1} for calculating the quantity y_{n+1}, it is necessary to calculate the kernel of the integral of the function $K(x,s,y)$ $(n+2)$ times, and for calculating the quantity y_n it is necessary to calculate the function $K(x,s,y)$ $(n+2)$ times. The method during of which the volume remains constant was constructed in (Imanova and Ibrahimov, 1998) and has the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) \quad (n = 0,1,2,...). \qquad (3)$$

Also, for defining the coefficients α_i, $\beta_i^{(j)}$ $(i, j = 0,1,2,...,k)$ a system of algebraic equations was found. In the paper (Mehdiyeva and Imanova, 1998), it was shown that method (3) may be obtained from the following finite-difference method:

$$\sum_{i=0}^{k} \alpha_i z_{n+i} = h\sum_{i=0}^{k} \beta_i z'_{n+i} \quad (n = 0,1,2,...). \qquad (4)$$

Taking into account that there are many papers (Dahlquist, 1956, 1959, Henrici, 1962, Enright, 1974, Brunner, 1970, Shura-Bura, 1952, Bakhalov, 1955, Godunov, 1977, Ibrahimov, 2002, Iserles, 1987, Lambert, 1973, Butcher, 1966, Huta, 1979, Kobza, 1925, Urabe, 1970, Sulitskiy, 1962), of the famous scientists devoted to the investigation of method (4) the relation between the methods (3) and (4) was found.

Notice that the accuracy of method (3) is bounded by the quantity $k+2$ (Mehdiyeva and Imanova, 1998). For increasing the accuracy of a multistep method of type (3) in § 1 a second derivative multistep method was suggested. In § 2 a method for defining its coefficients is stated.

§ 1. A second derivative finite-difference method. It is known that if by p we denote the degree of finite-difference method (4) the relation between its degree and order has the following form (the method's order quantity k is assumed to be known):

$$p \leqslant 2k.$$

They say that method (4) has the degree p if it holds:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih)) = O(h^{p+1}),\ h \to 0,$$

here p is an integer quantity.

If method (3) is stable, then $p \leqslant 2[k/2]+2$. Method (3) is stable if the roots of characteristical polynomial $\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i \lambda^i$ lie interior to a unit circle on whose boundary there are no multiple roots.

Denote by y'_m, y''_m the approximate, by $y'(x_m), y''(x_m)$ the exact values of derivatives of the function $y(x)$ at the points x_m.
Construct a more exact finite-difference method in the following form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i y''_{n+i}, \tag{1.1}$$

that is usually called a second derivative finite-difference method. Many papers have been devoted to the application of the suggested method (1.1) to the numerical solution of ordinary differential equations (Dahlquist, 1959, Ibrahimov, 2002 and Sulitskiy, 192).

Assume that method (1.1) has the degree p. Then for sufficiently smooth function $z(x)$ we can write:

$$\sum_{i=0}^{k}\alpha_i z(x+ih) - h\sum_{i=0}^{k}\beta_i z'(x+ih)$$

$$-h^2\sum_{i=0}^{k}\gamma_i z''(x+ih) = O(h^{p+1}), \quad h \to 0. \tag{1.2}$$

It is known that in method (1.1) the quantity k is an order of the correspondingly difference equation. Therefore, the accuracy of the finite-difference method is defined by means of the values of the degree of the method determined by asymptotic equalities (1.2).

The relation between the degree p and order k for method (1.1) in the general form is determined as follows:

$$p \leqslant 3k+1.$$

If method (1.1) is stable (definition of stability of method (1.1) coincides with definition of stability of method (4)), the relation between p and k has the following form (Henrici 1962, Iserles, 1987, Lambert, 1973, Butcher, 1966, Huta, 1979, Kobza, 1925, Urabe, 1970, Sulitskiy, 1962)

$$p \leqslant 2k+2 \tag{1.3}$$

and for any k there exist stable methods of type (1.1) with degree $p=2k+2$. Allowing for relation (1.3) we get that method (1.1) has a wider field of application than method (3). Therefore, here we consider application of method (1.1) to the numerical solution of equation (1). To this end, suppose that the function $K(x,s,z)$ continuous by aggregate of the arguments is defined in domain $G=(x_0 \leqslant s \leqslant x \leqslant X, |z| \leqslant Y)$, where it has continuous partial derivatives to $p+1$, the continuous function $f(x)$ is determined on the interval $[x_0, X]$ and at the same place it has continuous derivatives to some $p+1$, inclusively.

Since the solution of integral equation (1) the function $y(x)$ is continuous and defined on the interval $[x_0, X]$, where it has continuous derivatives to $p+1$, method (1.1) has the degree p, we can write:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2\gamma_i y''(x+ih)) = O(h^{p+1}), \quad h \to 0. \tag{1.4}$$

Assume that the solution of integral equation (1) was found by any method subject to which in (1) we get an identity. Then we can find $y'(x)$ and $y''(x)$, and as the result we have:

$$y'(x) = f'(x) + K(x,x,y(x)) + \int_{x_0}^{x} K'_x(x,s,y(s))ds, \tag{1.5}$$

$$y''(x) = f''(x) + \frac{d}{dx}K(x,x,y(x)) + \frac{\partial}{\partial x}K(x,s,y(s))\Big|_{s=x} + \int_{x_0}^{x} K''_{x^2}(x,s,y(s))ds. \tag{1.6}$$

Consider the difference $y(x_{n+k})-y(x_{n+k-1})$. Then we have:

$$y(x_{n+k})-y(x_{n+k-1})=f_{n+k}-f_{n+k-1}+\int_{x_0}^{x_{n+k-1}}(K(x_{n+k},s,y(s))$$

$$-K(x_{n+k-1},s,y(s)))ds+\int_{x_{n+k-1}}^{x_{n+k}}(K(x_{n+k},s,y(s))ds. \tag{1.7}$$

Applying the Taylor formula to the difference $K(x_m,s,y(s))-K(x_{m-1},\ s,y(s))$ and defining the integrals $\int_{x_0}^{x_{n+k-1}}(K'(x_{n+k-1},s,y(s))ds$ and $\int_0^{\xi_n}K''_{x^2}(\xi_{n+k},s,y(s))ds$ from (1.5) and (1.6), respectively, and taking into account (1.7) we get:

$$\begin{aligned}&y(x_{n+k})-y(x_{n+k-1})=f_{n+k}-f_{n+k-1}+hy'(x_{n+k-1})-hf'(x_{n+k-1})\\&-hK(x_{n+k-1},x_{n+k-1},y(x_{n+k-1}))+\frac{h^2}{2}y''(\xi_{n+k})-\frac{h^2}{2}f''(\xi_{n+k})\\&-\frac{h^2}{2}\frac{dK}{dx}\bigg|_{x=\xi_{n+k}}-\frac{h^2}{2}K'_x(\xi_{n+k})+\int_{x_{n+k-1}}^{x_{n+k}}K(x_{n+k},s,y(s))ds\\&-\frac{h^2}{2}\int_{x_{n+k-1}}^{\xi_{n+k}}K''_{x^2}(\xi_{n+k},s,y(s))ds,\end{aligned} \tag{1.8}$$

where $x_{n+k-1}<\xi_{n+k}<x_{n+k}$.

Using the Lagrange interpolation polynomial, we can write:

$$\frac{dK}{dx}\bigg|_{x=\xi_{n+k}}\approx\sum_{i=0}^{k}b_i\frac{dK}{dx}\bigg|_{x=x_{n+i}},\ K'_x(\xi_{n+k})\approx\sum_{i=0}^{k}l_iK'_x(x_{n+i}).$$

By means of some formula from theory of finite-difference method we can write the following one:

$$hy'(x_{n+k-1})\approx\sum_{i=0}^{k}\hat{\alpha}_iy_{n+i};\ h^2y''(x_{n+k})\approx\sum_{i=0}^{k}\bar{\alpha}_iy_{n+i}.$$

In order to calculate the higher accuracy integral we use the Hermitian formula. Then we have:

$$\int_{x_{n+k-1}}^{x_{n+k}}K(x_{n+k},s,y(s))ds$$

$$=h\sum_{i=0}^{k}\hat{\beta}_iK(x_{n+k},x_{n+i},y_{n+i})+h^2\sum_{i=0}^{k}\hat{\gamma}_iG(x_{n+k},x_{n+i},y_{n+i}).$$

Here

$$G(x,s,y(s))=\frac{\partial K(x,s,y(s))}{\partial s}=K'_s(x,s,y(s))+K'_y(x,s,y(s))y'.$$

Using the formula indicated above in (1.8) and making some transformations in it, we get the following generalized formula:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=\sum_{i=0}^{k}\alpha_i f_{n+i}+h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K(x_{n+j},x_{n+i},y_{n+i})$$

$$+h^2\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)}g(x_{n+i},x_{n+i},y_{n+i}), \qquad (1.9)$$

where $\alpha_i,\beta_i^{(j)},\gamma_i^{(j)}(i,j=0,1,...,k)$ are some real numbers expressed by the coefficients $b_i,l_i,\hat{\alpha}_i,\bar{\alpha}_i,\beta_i,\gamma_i\ (i=0,1,...,k)$:

$$g(x,s,y(s))=aK'_x(x,s,y(s))+bK'_s(x,s,y(s))+cK'_y(x,s,y(s))y'.$$

Notice that in future we'll consider only the case $a=b=c=1$.

Having applied method (1.1) to equation (1), we can get method (1.9). Indeed, assuming $x=x_{n+i}$ in (1.5) and (1.6) and taking into account (1.4) we have:

$$\sum_{i=0}^{k}\alpha_i y(x_{n+i})=\sum_{i=0}^{k}\alpha_i f_{n+i}+\sum_{i=0}^{k}\alpha_i\int_{x_0}^{x_n}K(x_{n+i},s,y(s))ds$$

$$+h\sum_{i=0}^{k}\beta_i K(x_{n+i},x_{n+i},y(x_{n+i}))+h^2\sum_{i=0}^{k}\gamma_i K'_x(x_{n+i},x_{n+i},y(x_{n+i}))$$

$$+h^2\sum_{i=0}^{k}\gamma_i\left.\frac{dK}{dx}\right|_{x=x_{n+i}}+h\sum_{i=0}^{k}\beta_i\int_{x_n}^{x_{n+i}}K'_x(x_{n+i},s,y(s))ds$$

$$+h^2\sum_{i=0}^{k}\gamma_i\int_{x_n}^{x_{n+i}}K''_{x^2}(x_{n+i},s,y(s))ds+O(h^{p+1}). \qquad (1.10)$$

Taking into account equation (1) in equality (1.10), we have:

$$\sum_{i=0}^{k}\alpha_i\int_{x_n}^{x_{n+i}}K(x_{n+i},s,y(s))ds+h\sum_{i=0}^{k}\beta_i\int_{x_n}^{x_{n+i}}K'_x(x_{n+i},s,y(s))ds$$

$$+h^2\sum_{i=0}^{k}\gamma_i\int_{x_n}^{x_{n+i}}K''_{x^2}(x_{n+i},s,y(s))ds$$

$$+h^2\sum_{i=0}^{k}\gamma_i K'_x(x_{n+i},x_{n+i},y(x_{n+i}))=h\sum_{i=0}^{k}\beta_i K(,x_{n+i},x_{n+i},y(x_{n+i}))$$

$$+h^2\sum_{i=0}^{k}\gamma_i K'_x(x_{n+i},x_{n+i},y(x_{n+i}))+O(h^{p+1}). \qquad (1.11)$$

It is easy to prove that it hold the following:

$$h\sum_{i=0}^{k}\beta_i K(x_{n+i},x_{n+i},y(x_{n+i}))+h^2\sum_{i=0}^{k}\gamma_i K'_x(x_{n+i},x_{n+i},y(x_{n+i}))$$
$$=h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K(x_{n+j},x_{n+i},y_{n+i})+O(h^{p+1}).$$

It is known that the coefficients $\beta_i^{(j)}\ (i,j=0,1,...,k)$ may be chosen so that it holds (Mehdiyeva and Imanova, 1998):

$$\sum_{i=0}^{k}\alpha_i(y(x_{n+i}-f_{n+i})$$
$$=h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K(x_{n+j},x_{n+i},y(x_{n+i}))+O(h^{p+1}). \qquad (1.12)$$

In (1.11) taking into account (1.12) and also the formulae suggested above with using some transformations, as the result we can obtain method (1.9).

Thus, we get that if method (1.1) has the degree p, the coefficients $\alpha_i,\beta_i^{(j)},\gamma_i^{(j)}\ (i,j=0,1,...,k)$ may be chosen so that the method obtained from (1.9) has the degree p.

Notice that for calculating y_{n+k} by method (1.9), it is required to determine the values of the function $y'(x)$ at the points $x_{n+i}(i=0,1,...,n)$, as well. For calculating these values we can use the method suggested by Sulitskiy (1962)

$$y'_{n+k}=\sum_{i=0}^{k-1}\beta_i y'_{n+i}+h^{-1}\sum_{i=0}^{k}\alpha'_i y_{n+i}.$$

Here we suggest the following method that is obtained by applying the method by Mehdiyeva and Imanova (1998) to equation (1.5)

$$\sum_{i=0}^{k}\alpha_i y'_{n+i}=h\sum_{j=0}^{k}\sum_{i=0}^{k}\alpha_i(f'_{n+i}+K(x_{n+i},x_{n+i},y_{n+i})$$
$$+h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K'_x(x_{n+j},x_{n+i},y_{n+i}). \qquad (1.14)$$

Notice that if $K(x,s,y)$ is independent of x from (1.14) we get that $y'(x)=f'(x)+K(x,y(x))$ and in this case it is not necessary to use method (1.14). It is clear that the accuracy of method (1.9) depends on the values of its coefficients $\alpha_i,\beta_i^{(j)},\gamma_i^{(j)}\ (i,j=0,1,2,...,k)$. Therefore, define them. This method has some advantage and deficiencies. One of the advantages is constant amount of calculations of the functions $K(x,z,y)$ and $g(x,z,y)$ at each step. The main deficiency of the method is calculation of the function $g(x,z,y)$ at each step, since therewith there arises necessity to calculate the functions K'_x,K'_z,K'_y and

y' that from the point of view of calculations may be equivalent to the calculation of the function $K(x,z,y)$. However, taking into account possibility of contemporary computers, some times we can neglect the indicated deficiency.

2. Finding coefficients in second derivative finite-difference method

Usually, the name of numerical methods agrees with the scheme of finding its coefficients. In this connection the finite-difference and multistep methods and also Obreshkov k-step methods are constructed by the same formulae. Here, for determining the coefficients of method (1.9) we'll use expansion of functions in Taylor's series and finite-difference equations. Therefore, we call the construction of the method by the described scheme a finite-difference method. To this end, we consider a special case and assume that $K(x,s,y) = F(x,y)$ Then equation (1) is written in the form:

$$y(x) = f(x) + \int_{x_0}^{x} F(s, y(s))ds. \qquad (2.1)$$

Apply method (1.9) to equation (2.1) and have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i f_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)} F(x_{n+i}, y_{n+i}) + h^2\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)} G`(x_{n+i}, y_{n+i}). \qquad (2.2)$$

Here $G(x,y) = F'_x(x,y) + F'_y(x,y)y'$.

If we denote

$$\beta_i = \sum_{j=0}^{k}\beta_i^{(j)},\ \gamma_i = \sum_{j=0}^{k}\gamma_i^{(j)}, \qquad (2.3)$$

the method (2.2)may be rewritten in the form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i f_{n+i} + h\sum_{i=0}^{k}\beta_i F_{n+i} + h^2\sum_{i=0}^{k}\gamma_i G_{n+i}, \qquad (2.4)$$

where $F_m = F(x_m, y_m), G_m = G(x_m, y_m)\,(m = 0,1,2,...)$ taking into account smoothness of the function $f(x)$, we can write

$$\sum_{i=0}^{k}\alpha_i f(x_{n+i}) = h\sum_{i=0}^{k}\beta_i f'(x_{n+i}) + h^2\sum_{i=0}^{k}\gamma_i f''(x_{n+i}) + O(h^{p+1}), \quad h \to 0.$$

After taking into account the obtained one in (2.4) and replacing y_m, y'_m and y''_m by their exact values $y(x_m)$, $y'(x_m)$ and $y''(x_m)$, relation (2.4) will have the following form:

$$\sum_{i=0}^{k}\alpha_i y(x_{n+i}) = h\sum_{i=0}^{k}\beta_i (f'(x_{n+i}) + F(x_{n+i}, y(x_{n+i}))) + h^2\sum_{i=0}^{k}\gamma_i (f''(x_{n+i}) + F'_x(x_{n+i}, y(x_{n+i})) + F'_y(x_{n+i}, y(x_{n+i}))y'(x_{n+i})) + O(h^{p+1}), \qquad h \to 0.$$

Hence we get:

$$\sum_{i=0}^{k}\alpha_i y(x_{n+i}) = h\sum_{i=0}^{k}\beta_i y'(x_{n+i}) + h^2\sum_{i=0}^{k}\gamma_i y''(x_{n+i}) + O(h^{p+1}), \quad h\to 0. \tag{2.5}$$

Consider the following expansion

$$y(x+ih) = y(x) + hy'(x) + ... + h^p y^{(p)}(x)/p! + O(h^{p+1}),$$

$$y^{(\nu)}(x+ih) = y^{(\nu)}(x) + hy^{(\nu+1)}(x) + \ldots + h^{p-\nu}y^{(p)}(x)/(p-\nu)! + O(h^{p+1-\nu}) \ \ (\nu = 1,2).$$

Take these expansions into account in (2.5) and have:

$$\sum_{i=0}^{k}\alpha_i\left(\sum_{j=0}^{p}h^j y^{(j)}(x)/j!\right) - h\sum_{i=0}^{k}\beta_i\left(\sum_{j=0}^{p-1}h^j y^{(j+1)}(x)/j!\right)$$
$$-h^2\sum_{i=0}^{k}\gamma_i\left(\sum_{j=0}^{p-2}h^j y^{(j+2)}(x)/j!\right) + O(h^{p+1}), (0! = 1). \tag{2.6}$$

Taking into account linear independence of the system $y(x), hy'(x), ...,$ $h^p y^{(p)}(x)$, from (2.6) we get that in order method (2.4) have the degree p, then

$$\sum_{i=0}^{k}\alpha_i = 0,\ \sum_{i=0}^{k}\alpha_i = \sum_{i=0}^{k}\beta_i,$$
$$\sum_{i=0}^{k}\frac{i^l}{l!}\alpha_i = \sum_{i=0}^{k}\frac{i^{l-1}}{(l-1)!}\beta_i + \sum_{i=0}^{k}\frac{i^{l-2}}{(l-2)!}\gamma_i = 0 \qquad (l = 2,3,...,p) \tag{2.7}$$

is a necessary and sufficient condition.

Thus, for determining the coefficients $\alpha_i, \beta_i, \gamma_i \, (i = 0,1,2,...,k)$ we get a homogeneous system of linear-algebraic equation in which the amount of unknowns equals $3k+3$, the amount of equation $p+1$. It is known that in order the homogeneous system of linear equations have a nontrivial solution (non zero) there should be $p+1 < 3k+3$.

Consequently, between the degree and order of method (2.4) there is the following relation:

$$p \leqslant 3k+1.$$

For any k there exists a method with degree $p = 3k+1$. We can prove that the method with degree $p = 3k+1$ of type (2.4) is not unique. This is connected with the fact that the coefficients $\beta_i^{(j)}, \gamma_i^{(j)} \, (i,j = 0,1,2,...,k)$ of method (1.9) are determined from system (2.3) but not from system (2.7) as the coefficients of method (2.4). However, it is seen from system (2.3) that the coefficients $\beta_i^{(j)}, \gamma_i^{(j)} \, (i,j = 0,1,2,...,k)$ of method (1.9) are determined by means of the coefficients $\beta_i, \gamma_i \, (i = 0,1,2,...,k)$ of method (2.4). As is known, in solving practical problems, the stable methods are more interesting. For establishing the stability of method (1.9) we'll use the following definition.

Definition. Multistep method (1.9) with constant coefficients is said to be stable if the roots of its characteristically polynomial $\rho(\lambda)=\sum_{i=0}^{k}\alpha_i\lambda^i$ lie interior to a unit circle on whose boundaries there are no multiple roots.

If method (1.9) is stable and the degree p, then $p \leqslant 2k+2$. Indeed, since the stable method of type (2.4) has the degree $p \leqslant 2k+2$, the coefficients in the linear part of method (2.4) and (1.9) coincide, and stability of these methods depend on the coefficients $\alpha_i (i=0,1,2,...,k)$ in the linear part of these methods. Therefore, maximal values of stable methods obtained from methods (2.4) and (1.9) will coincide.

Notice that the amount of arithmetic operations during solving algebraic equation (2.7) are measured by means of the following function:

$$a_0(p+1)^3+a_1(p+1)^2+a_2(p+1)+a_0.$$

For decreasing the amount f algebraic operations during solving system (2.7), here we get suggest a scheme that may be called a generalization of the scheme by Urabe (1970). Before stating this scheme we remind the conditions to which the coefficients of finite-difference method (2.4) should satisfy:

A. The coefficients α_i, β_i, γ_i $(i=0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \neq 0$.

B. The polynomials

$$\rho(\lambda)\equiv\sum_{i=0}^{k}\alpha_i\lambda^i,\ \vartheta(\lambda)\equiv\sum_{i=0}^{k}\beta_i\lambda^i,\ \gamma(\lambda)\equiv\sum_{i=0}^{k}\gamma_i\lambda^i,$$

have no common multipliers differ from constants.

C. $\vartheta(1)\neq 0,\ \rho \geqslant 1$.

By means of the shift operator $E\ (Ey(x)=y(x+h))$, re rewrite method (2.4) in the following form: $\rho(E)y_n - h\vartheta(E)y'_n - h^2\vartheta(E)y''_n = 0.$

Urabe (1970) invetigated method (2.4) under analyticity of the function $y(x)$ and $k=2$. Here we assume that the continuous function $y(x)$ is determined on $[x_0, X]$ and at the same place it ha continuous derivatives to $p+1$, inclusively. Then taking into account that method (2.4) has the degree p, we can write:

$$\rho(E)y(x_n)-h\vartheta(E)y'(x_n)-h^2\vartheta(E)y''(x_n)$$
$$=h^{p+1}\sum_{j=1}^{k}(c_j y^{(p+1)}(\xi_j^{(1)})+b_j y^{(p+1)}(\xi_j^{(2)})+d_j y^{(p+2)}(\xi_j^{(3)})), \qquad (2.8)$$

where $\xi_j^{(m)} \in (x_n, x_{n+j})\ (m = 1,2,3; j = 1,2,...,k)$,

$\xi_j^{(m)} \in (x_n, x_{n+j})\ (m = 1,2,3;\ j = 1,2,...,k),$ are some real numbers. By means of the following linear operator $E_{t_i} y(x) = y(x + t_i h)$ we can rewrite the right hand side of equation (2.8) in the form:

$$h^{p+1}\sum_{j=1}^{k}(c_j y^{(p+1)}(\xi_j^{(1)}) + b_j y^{(p+1)}(\xi_j^{(2)}) + d_j y^{(p+1)}(\xi_j^{(3)}))$$
$$= h^{p+1}\sum_{j=1}^{k}(c_j E_{l_j^{(1)}} + l_j E_{l_j^{(2)}} + d_j E_{l_j^{(3)}}) y^{(p+1)}_{(x_n)}, h \to 0. \quad (2.9)$$

Here, the quantities $l_j^{(m)}$ satisfy the condition $0 < l_j^{(m)} < j\,(m = 1,2,3; j = 1,2,...,k)$. Taking into account that $y^{(p+1)}(x)$ is bounded on $[x_0, X]$ and using the differentiation operator D, we have:

$$\rho(E)y(x_n) - h\vartheta(E)Dy(x_n) - h^2\gamma(E)D^2 y(x_n)$$
$$\sim C(hD)^{p+1} y(x_n),\ \ h \to 0. \quad (2.10)$$

Using the substitution $\tau = \exp(hD)$, we rewrite relation (2.10) in the following form:

$$\rho(\tau) - \vartheta(\tau)\ln\tau - \gamma(\tau)(\ln\tau)^2 \sim C(\ln\tau)^{p+1},\ \tau \to 1. \quad (2.11)$$

If we assume that method (2.4) converges, then it follows from (2.11) that

$$\rho(1) = 0 \quad (2.12)$$

that usually is called a necessary condition of convergence . Hence it follows that $\lambda = 1$ is a root of the polynomial $\rho(\lambda)$. Therefore, using the substitution $\xi = \tau - 1$, we rewrite relation (2.11) in the form:

$$\rho(1+\xi)(\ln(1+\xi))^{-1} - \vartheta(1+\xi) - \gamma(1+\xi)\ln(1+\xi) = O(\xi^p), \xi \to 0, \quad (2.13)$$

where

$$\rho(1+\xi) \equiv \sum_{j=0}^{k-1}\rho_j^{(1)}\xi^{j+1},\ \vartheta(1+\xi) \equiv \sum_{j=0}^{k}\rho_j^{(2)}\xi^j,\ \gamma(1+\xi) \equiv \sum_{j=0}^{k}\rho_j^{(3)}\xi^j.$$

Using the expansion of the function $\xi(\ln(1+\xi))^{-1}$ and $\ln(1+\xi)$ in (2.13) for finding the coefficients of polynomials $\rho_j^{(m)}\,(j = 0,1,2,...; m = 1,2,3)$, we get the following system of equations:

$$\sum_{i=0}^{j}\bar{c}_i\,\rho_{j-1}^{(1)} + \sum_{i=0}^{j}(-1)^{j-i+1}\rho_{i-1}^{(3)}/(j-i+1) = \rho_j^{(2)}(j = 0,1,2,...,k; \rho_k^{(1)} = 0),$$
$$\sum_{i=j-k+1}^{j}\bar{c}_i\,\rho_{j-1}^{(1)} + \sum_{i=j-k}^{j}(-1)^i\rho_{j-1}^{(3)}/i = 0\ (j = k+1, k+2,..., p-1). \quad (2.14)$$

where

$$\hat{c}_\nu = \frac{1}{\nu!}\int_0^1 s(s-1)...(s-\nu+1)ds \ (\nu = 0,1,2,...).$$

Here the coefficients β_i, $\gamma_i \ (i = 0,1,2,...,k)$ are calculated by means of the following recurrent relations:

$$\alpha_0 = -\rho_0^{(1)} + \rho_1^{(1)} - \rho_2^{(1)} + ... + (-1)^{k-1}\rho_{k-2}^{(1)} + (-1)^k \rho_{k-1}^{(1)},$$

$$\alpha_i = \sum_{j=i-1}^{k-1}(-1)^{j-i+1}(j+1)j(j-1)...(j-i+2)\rho_j^{(1)} / i! \ (i = 1,2,...,k),$$

$$\beta_0 = \rho_0^{(2)} - \rho_1^{(2)} + \rho_2^{(2)} + ... + (-1)^{k-1}\rho_{k-2}^{(2)} + (-1)^k \rho_{k-1}^{(2)},$$

$$\beta_i = \sum_{j=i}^{k}(-1)^{j-i} j(j-1)...(j-i+1)\rho_j^{(2)} / i! \ \ (i = 1,2,...,k). \tag{2.15}$$

For calculating $\gamma_i \ (i = 0,1,2,...,k)$, it suffices in (2.15) to substitute β_i for γ_i and $\rho_i^{(2)}$ for $\rho_i^{(3)} \ (i = 0,1,2,...,k)$.

Thus, for finding the coefficients $\beta_i, \gamma_i \ (i = 0,1,2,...,k)$ we get the system of linear algebraic equation (2.14) and the system of recurrent equations (2.15). it is easy to show that the amount of arithmetical operation while investigating systems (2.14) and (2.15) will be measured by means of the following function: $a_0(p-k)^3 + a_1(p-k)^2 + a_2(p-k) + a_3 + 3k^2$.

For calculating the coefficients $\hat{c}_\nu \ (\nu = 0,1,2,...)$ here we suggest the following recurrent relation: $\hat{c}_m = \sum_{i=1}^{m}(-1)^{i-1}\hat{c}_{m-i} / (i+1) \ (\hat{c}_0 = 1, m = 1,2,...)$.

If we consider the case $\gamma_i = 0 \ (i = 0,1,2,...,k)$, then from (2.5) we get a finite-difference method of type (3). In this case, the system of equations (2.14) consists of one equation and therefore the amount of arithmetic operations for calculating α_i, $\beta_i \ (i = 0,1,2,...,k)$ coefficients of method (3) is measured by means of the function $3k^2$. Thus, we obtained that for finding the coefficients of method (3) and (2.5) it is desirable to use system (2.14) and (2.15).

Notice, that for finding solution of nonlinear integral equation one can use the forward-jumping methods(Imanova et al, 2010).

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