New Fuzzy Entropy Measure of Order α

Mohammad Al-Talib
Department of Statistics , Yarmouk University, 21163 Irbid Jordan
m.altalib@yu.edu.jo

Amjad Al-Nasser
Department of Statistics , Yarmouk University, 21163 Irbid Jordan
Amjadn@yu.edu.jo

Abstract

In this article, a new fuzzy entropy measure of order αis proposed. The fuzzy entropy axiomatic requirements are discussed for the new measure, and an empirical comparison are made with several entropy measures including Shannon, Rényiand Kapur. It turns out, the proposed measure satisfies all axiomatic and outperform the other entropy measures in terms of the fuzziness degree.

Keywords: Fuzzy sets; Fuzzy entropy; Entropy axiomatic; Shannon entropy.

1. Introduction

The novelty of fuzzy sets (FS) dated back to Zadeh (1965). The main idea of FS is to model non-statistical vague phenomena. Since then, the theory of FS became an interesting research area in many scientific disciplines including but not limited to; engineering, image processing, data mining, medical science, clustering, information technology and statistical information theory. Fuzziness as a feature of uncertainty can be explained as a result of a given decision on an event that to be considered as a member of a set or not. In such cases, the event is considered as a fuzzy rather than sharply defined as collection of points (Zadeh, 1968). Fuzzy entropy (FE) is the measure of vagueness and ambiguity of uncertainties; it isused as a measure of such fuzziness (Bhat and Baig, 2017).The information theory throughout the principle of entropy has been used widely in fuzzy set theory because of its capability in dealing with a lack of information models. Several FE measures has been discussed in the literature. De-Luca and Termini (1972) proposed the first entropy extension of Shannon (1948) entropy; by suggesting a nonprobability FE, also they defined the basic properties of the proposed FE as sharpness, maximality, resolutions and symmetry; which considered as a road map for developing any new FE measure. Then after, several authors introduced modified FE measures (Ohlan, 2015; Naidu et al, 2017; Zhang et al, 2012; Al-Sharhan et al.2001; Bhandari and Pal, 1993; Kapur, 1997;Parkash and Sharma, 2002). On another point, some application of entropy in goodness of fit tests for non-fuzzy datasetsare possible to be generalized to the fuzzy entropy. (Zamanzadeand Mahdizadeh,(2016, 2017); Zamanzade andArghami, 2011;Zamanzade, 2014) as other applications were generalized to the fuzzy sets and fuzzy entropyin different fields, such as sampling (see; Greenfield, 2012;Cetintav, 2016), goodness of fit (see; Grzegorzewski and Szymanowski, 2014;Eliason and Stryker, 2009), testing (see; Xie, 2010), and many other fields.

In this article, we are proposing a new entropy measure of order alpha. The article is organized as follows: in Section 2the basic concepts of FE are discussed. The proposed entropy along with its properties are given in section 3. Empirical comparison between several FE measures is given in section 4, to study the performance of the proposed FE.The article ends with some concluding remarks in section 5.

2. Basic Concepts of Fuzzy Entropy

The theory of fuzzy sets was firstly presented byZadeh (1965);let *A*denote a fuzzy set such that $A = \{x_i\ , i = 1, \dots, n\}$, with a characteristic function $\mu_{A(x_i)}$ describes the degree of belonging for an element *x* to *A*. Based on this function, the concept of fuzzy entropy is introduced by De Luca and Termini (1972), the suggested the following fuzzy entropy

$$H(A) = -K\sum_{i=1}^{n}[\mu_{A(x_i)}\log\mu_{A(x_i)} + (1-\mu_{A(x_i)})\log(1-\mu_{A(x_i)})],$$

where $\mu_{A(x_i)}$ is the ith membership function and *K*is a constant equal to 1/n;alongside entropy measure should satisfied the following set of axiomatic requirements as given by De Luca and Termini (1972):

Axiom 1. *Sharpness*: $H(\mu_{A(x_i)}) = 0$for each *i*; if and only if$\mu_{A(x_i)}$ is a crisp set. That is to say$\mu_{A(x_i)} = 0\ or\ 1$

Axiom 2.*Maximality*: $H(\mu_{A(x_i)})$ has a unique maximum value if $\mu_{A(x_i)} = \frac{1}{2}.\forall i$

Axiom 3.*Resolutions*: $H(\mu_{A(x_i)}{}^{*}) \le H(\mu_{A(x_i)})$ where $\mu_{A(x_i)}{}^{*}$ is crisper than$\mu_{A(x_i)}$

Axiom 4. *Symmetry*: $H(\mu_{A(x_i)}) = H(1-\mu_{A(x_i)})\forall\ i.$

Later on, several generalized fuzzy entropies surfaced through the years in the literature, important FE measures corresponding to Rényi'sentropy proposed byBhandari and Pal (1993);

$$H_\alpha(A) = (1-\alpha)^{-1}\sum_{i=1}^{n}\log[\mu_{A(x_i)}^{\alpha} + (1-\mu_{A(x_i)})^{\alpha}];\ \alpha \ne 1. \alpha > 0.$$

Moreover, additional entropy measure that defined byKapur(1997)

$$H_\alpha(A) = (1-\alpha)^{-1}\sum_{i=1}^{n}[\{\mu_{A(x_i)}^{\alpha} + (1-\mu_{A(x_i)})^{\alpha}\} - 1];\ \alpha \ne 1, \alpha > 0.$$

These measures among many others developed the generalized fuzzy entropy measures, interesting applications and related entropy measures can be found in Gurdialet. al.(2001), Pal and Bezdek(1994), Hu and Yu (2004), Priti, and Sheoran (2014), Bajaj and Hooda (2010).

3. Proposed Fuzzy Entropy Measure

In this section, a new parametric fuzzy entropy measure is proposed. The suggested mathematical formula of the new FE is given as:

$$H_\alpha^{NT}(\mu_{A(x_i)}) = \sum_i \left(\frac{\mu_{A(x_i)}{}^{\frac{\alpha}{2}}\left(1-\mu_{A(x_i)}\right)^{\frac{\alpha}{2}}}{\mu_{A(x_i)}e^{-\alpha\left(1-\mu_{A(x_i)}\right)} + \left(1-\mu_{A(x_i)}\right)e^{-\alpha\mu_{A(x_i)}}}\right)^{\frac{1}{\alpha}}\ ;\ \alpha > 0\ , \tag{1}$$

where $\{\ 0 \le \mu_{A(x_i)} \le 1; i = 1.2.\cdots.n\ \}$ fuzzy memberships, and*n* is the total number of memberships. It is a valid measure that satisfies all entropy function properties.

Theorem 1. The measure given in (1) is satisfies all axiomatic requirements of the FE.

Proof: We will prove that the axioms (1) to (4) are satisfied by equation (1).

***Axiom 1.* Sharpness:**
Replacing the value of $\mu_{A(x_i)} = 0\ or\ 1$ in the numerator of (1) then $H_\alpha^{NT}\left(\mu_{A(x_i)}\right) = 0$. This result is a straight forward $\forall\ \alpha > 0$. Conversely, assume that $H_\alpha^{NT}\left(\mu_{A(x_i)}\right) = 0$, then

$$\mu_{A(x_i)}{}^{\frac{\alpha}{2}}\left(1 - \mu_{A(x_i)}\right)^{\frac{\alpha}{2}} = 0, \forall > 0$$

then $\mu_{A(x_i)} = 0\ or\ 1$.

***Axiom 2.*Maximality**
By differentiating (1) with respect to $\mu_{A(x_i)}$,we get

$$\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}} = \left(\frac{e^{\alpha\left(1+\mu_{A(x_i)}\right)}\left[-\mu_{A(x_i)}\left(1-\mu_{A(x_i)}\right)\right]^{\frac{\alpha}{2}}}{e^\alpha - \mu_{A(x_i)}e^\alpha + \mu_{A(x_i)}e^{2\alpha\mu_{A(x_i)}}}\right)^{\frac{1}{\alpha}} \times$$
$$\frac{\mu_{A(x_i)}e^{2\alpha\mu_{A(x_i)}}\left[2(\mu_{A(x_i)}-1) + \alpha\left(1 + 2\mu_{A(x_i)}(\mu_{A(x_i)}-2)\right)\right] + (\mu_{A(x_i)}-1)e^\alpha\left[\alpha\left(2\mu_{A(x_i)}{}^2 - 1\right) - 2\mu_{A(x_i)}\right]}{2\alpha\mu_{A(x_i)}(\mu_{A(x_i)}-1)\left[e^\alpha\left(\mu_{A(x_i)}-1\right) - \mu_{A(x_i)}e^{2\alpha\mu_{A(x_i)}}\right]}$$

Then solving this derivative by setting $\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}} = 0$; we observe that $\mu_{A(x_i)} = 0.5$.

Now, assume that $0 < \mu_{A(x_i)} < 0.5$, then

$$\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}} > 0\ \text{, for}\ \alpha > 0.$$

Also,when$0.5 < \mu_{A(x_i)} < 1$, then

$$\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}} < 0\ \text{, for}\ \alpha > 0.$$

and for$\mu_{A(x_i)} = 0.5$, then

$$\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}} = 0, \text{for}\ \alpha > 0.$$

Thus,$\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}}$is a concave function which has a global maximum at$\mu_{A(x_i)} = 0.5$. Hence,$H_\alpha^{NT}\left(\mu_{A(x_i)}\right)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_{A(x_i)} = 0.5, \forall i = 1, \dots, n$, which proofs the uniqueness of the maximum value, for a graphical visualization, $\frac{\partial H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}}$is graphed for different values of α.

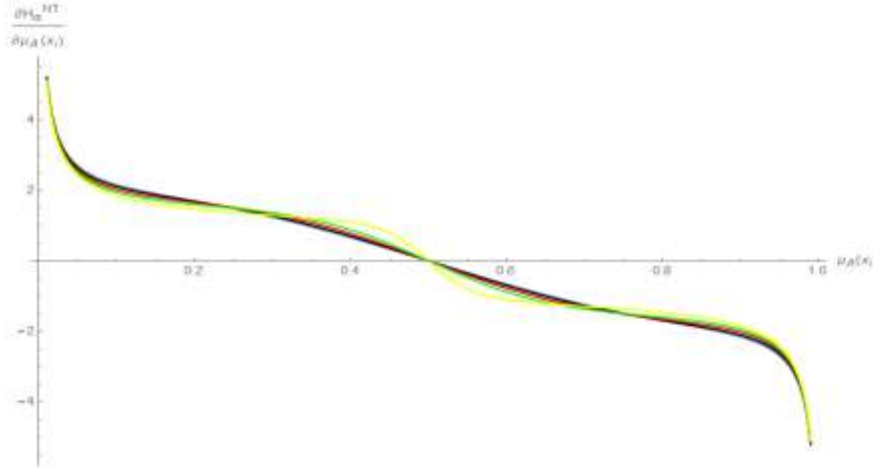

Figure 1. First derivative at different values of α. (Blue line: $\alpha = 0.2$, Black: $\alpha = 0.8$,Red: $\alpha = 2$, Green: $\alpha = 4$ and Yellow: $\alpha = 16$).

Moreover, evaluating the second derivative of (1), we have

$$\frac{\partial^2 H_\alpha^{NT}\left(\mu_{A(x_i)}\right)}{\partial \mu_{A(x_i)}{}^2} = \frac{1}{4\alpha^2 \mu_{A(x_i)}{}^2 \left(\mu_{A(x_i)} - 1\right)^2 \left(\left(\mu_{A(x_i)} - 1\right) e^{\alpha} - \mu_{A(x_i)} e^{2\alpha \mu_{A(x_i)}}\right)^2}$$

$$\times \left(\frac{e^{\alpha + \alpha \mu_{A(x_i)}} \left(-\mu_{A(x_i)} \left(\mu_{A(x_i)} - 1\right)\right)^{\alpha/2}}{e^{\alpha} - \mu_{A(x_i)} e^{\alpha} + \mu_{A(x_i)} e^{2\alpha \mu_{A(x_i)}}} \right)^{\frac{1}{\alpha}} \Big[2\mu_{A(x_i)} e^{\alpha + 2\alpha \mu_{A(x_i)}} (\mu_{A(x_i)} - 1)(2\alpha$$

$$- 4\mu_{A(x_i)}(\mu_{A(x_i)} - 1) + 8\alpha^3 \mu_{A(x_i)}{}^2 \left(\mu_{A(x_i)} - 1\right)^2 + \alpha^2 (1 + 4(\mu_{A(x_i)} - 2)(\mu_{A(x_i)}$$

$$- 1)\mu_{A(x_i)}(1 + \mu_{A(x_i)}))) + e^{4\alpha \mu_{A(x_i)}} \mu_{A(x_i)}{}^2 (4\left(\mu_{A(x_i)} - 1\right)^2 + 4\alpha \mu_{A(x_i)}(\mu_{A(x_i)}$$

$$- 1)(2\mu_{A(x_i)} - 3) + \alpha^2(-1 + 4(\mu_{A(x_i)} - 1)\mu_{A(x_i)}(1 + \mu_{A(x_i)}(\mu_{A(x_i)} - 3))))$$

$$+ e^{2\alpha}(\mu_{A(x_i)} - 1)^2 (4\mu_{A(x_i)}{}^2 + 4\alpha \mu_{A(x_i)}(1 + \mu_{A(x_i)} - 2\mu_{A(x_i)}{}^2) + \alpha^2(-1 + 4\mu_{A(x_i)}(1$$

$$- 2\mu_{A(x_i)} + \mu_{A(x_i)}{}^3))) \Big].$$

The value of $\frac{\partial^2 H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}{}^2}$ when $\mu_{A(x_i)}$ =0.5 is,

$$c_\alpha \cdot (-32\alpha^2 e^{2\alpha} - 4\alpha^3 e^{2\alpha}), \text{ where } c_\alpha \cong \frac{\left(e^{-0.19\alpha}\right)^{1/\alpha}}{4\alpha^2 e^{2\alpha}}, \alpha > 0,$$

thus, we see that the maximum value of the proposed fuzzy entropy exists at $\mu_{A(x_i)}$ =0.5. Figure 2 present avisualization of $\frac{\partial^2 H_\alpha^{NT}(\mu_{A(x_i)})}{\partial \mu_{A(x_i)}{}^2}$at different values of α.

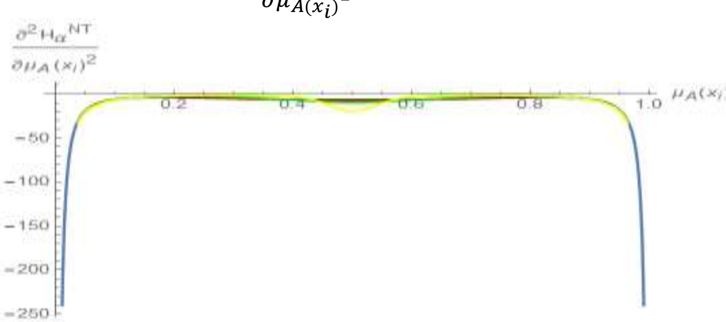

Figure 2.Second derivative at different values of α. (Blue line: $\alpha = 0.2$, Black: $\alpha = 0.8$, Red: $\alpha = 2$, Green: $\alpha = 4$ and Yellow: $\alpha = 16$).

where the maximum value of (1) at 0.5 is equal to 0.5Exp(0.5) =.8243.Table 1gives the numerical results of the measure for several values of the membership $\mu_{A(x_i)}$ and for α =0.1, 0.7, 2, 4 and 16. It could be noted that, as α increasing the entropy value is decreases within the same membership value. However, at any given α the proposed entropy measure H_α^{NT} is increasing in the interval $0 \leq \mu_{A(x_i)} \leq \frac{1}{2}$ and decreasing in the interval $\frac{1}{2} \leq \mu_{A(x_i)} \leq 1$.

Table 1: Different values of H_α^{NT}

$\mu_{A(x_i)}$	α	H_α^{NT}	α	H_α^{NT}	α	H_α^{NT}	α	H_α^{NT}	α	H_α^{NT}
0		0.0000		0.0000		0.0000		0.0000		0.0000
.1		0.3581		0.3529		0.3463		0.3400		0.3337
.2		0.5492		0.5407		0.5267		0.5137		0.4954
.3		0.6962		0.6896		0.6770		0.6623		0.6324
.4		0.7913		0.7890		0.7843		0.7777		0.7532
.5	0.1	0.8243	0.7	0.8243	2	0.8243	4	0.8243	16	0.8243
.6		0.7913		0.7890		0.7843		0.7777		0.7532
.7		0.6962		0.6896		0.6770		0.6623		0.6324
.8		0.5492		0.5407		0.5267		0.5137		0.4954
.9		0.3581		0.3529		0.3463		0.3400		0.3337
1.0		0.0000		0.0000		0.0000		0.0000		0.0000

***Axiom3.* Resolutions** As discussed before, the measure $H_\alpha(\mu_{A(x_i)})$ is monotonically increasing function for $0 \leq \mu_{A(x_i)} \leq \frac{1}{2}$; and is monotonically decreasing function for $\frac{1}{2} \leq \mu_{A(x_i)} \leq 1$. Therefore, it is concave function. Hence, $H_\alpha^{NT}({\mu_{A(x_i)}}^*) \leq H_\alpha^{NT}(\mu_{A(x_i)})$ where ${\mu_{A(x_i)}}^*$ is sharpened version of $\mu_{A(x_i)}$. For better visualization see Figure 3.

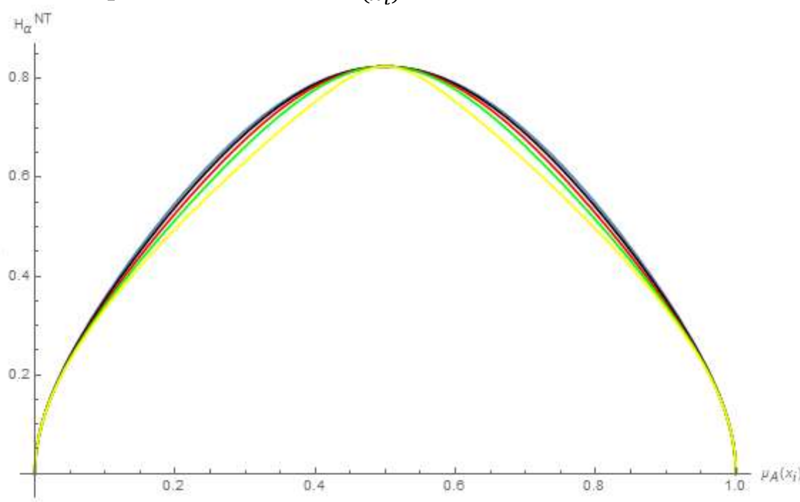

Figure3. $H_\alpha(\mu_{A(x_i)})$at different values of α. (Blue line: $\alpha = 0.2$, Black: $\alpha = 0.8$, Red: $\alpha = 2$, Green: $\alpha = 4$ and Yellow: $\alpha = 16$).

***Axiom 4.*Symmetry** By substituting $1 - \mu_{A(x_i)}$ instead of $\mu_{A(x_i)}$in Equation (1),we get that$H(\mu_{A(x_i)}) = H(1 - \mu_{A(x_i)}), \forall i = 1, \dots, n$.

And hence, Theorem 1 is proven.

4. Comparative Examples

To study the performance of the proposed fuzzy entropy, a numerical comparison between the proposed FE measure and some existence measures: De Luca and Termini measure, Bhandari and Pal measureand Kapur are included. The results of these comparisons are given in Table 3.

Table 2: Comparison $H_\alpha^{NT}$with different entropy measures when $\alpha = 2$

$\mu_{A(x_i)}$	H_α^{NT}	De Luca and Termini	Bhandari and Pal	Kapur
0	0.0000	0.0000	0.0000	0.0000
.1	0.3463	0.3250	0.1984	0.1800
.2	0.5267	0.5004	0.3856	0.3200
.3	0.6770	0.6108	0.5447	0.4200
.4	0.7843	0.6730	0.6539	0.4800
.5	0.8243	0.6931	0.6931	0.5000
.6	0.7843	0.6730	0.6539	0.4800
.7	0.6770	0.6108	0.5447	0.4200
.8	0.5267	0.5004	0.3856	0.3200
.9	0.3463	0.3250	0.1984	0.1800
1.0	0.0000	0.0000	0.0000	0.0000
Total	5.4959	4.9115	4.2583	3.3
Normalized Entropy	0.4996	0.4465	0.3841	0.3

The results indicate that the proposed measure is more informative than the other entropy measures. This result is clear that within all membership values H_α^{NT}provides a small uncertainty value as the entropy is the largest (Tomar and Ohlan, 2014), in which we can rank these entropies measures in one and only one manner:

$$H_\alpha^{NT} > De\ Luca\ and\ Termini\ > Bhandari\ and\ Pal\ > Kapur$$

This even true with different values of α.

5. Concluding Remarks

In this paper, a new FE of order α is proposed. The proposed FE measure of information is valid and satisfies all FE axiomatic requirements. Comparisons with other entropies have been made through numerical computation of the normalized entropy value. The numerical results indicated that the proposed FE measure outperform the existing measures in terms of the informative degree. This resultencouraged us to generalize the proposed FE measure in the upcoming future research.

Another possible topic for future research is to use entropy of orderα in fuzzy setting for multi criteria decision making problems which has an application in evaluating entrepreneurial opportunities, for more on this interesting topic, see Adel Rastkhiz, 2018.

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