The Exponentiated Generalized Topp Leone-G Family of Distributions: Properties and Applications

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Abstract

In this paper, we propose a new class of continuous distributions called the exponentiated generalized Topp Leone-G family that extends the Topp Leone-G family introduced by Al-Shomrani et al. (2016). We derive explicit expressions for certain mathematical properties of the new family such as; ordinary and incomplete moments, generating functions, reliability analysis, Lorenz and Bonferroni curves, Rényi entropy, stress strength model, moment of residual and reversed residual life, order statistics, extreme values and characterizations. We discuss the maximum likelihood estimates and the observed information matrix for the model parameters. Two real data sets are used to illustrate the flexibility of the new family.

Keywords: Exponentiated generalized-G family; Maximum likelihood estimation; Moments; Order statistics, Topp leone-G family.

1. Introduction

There has be an increase interest in constructing new generated families of univariate continuous distributions by adding additional shape parameter(s) to a baseline model due to the desirable properties of the new models. Some of the well-known generated families are the following: exponentiated-G by Gupta et al. (1998), beta-G by Eugene et al.(2002), Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G by Alexander et al. (2012), logistic-G by Torabi and Montazari (2014), Lomax-G by Cordeiro et al. (2014), Kumaraswamy Marshall-Olkin-G by Alizadeh et al. (2015), odd-Burr generalized-G by Alizadeh et al. (2016), beta weibull-G by Yousof et al. (2017), among others.

Let $g(x;\phi)$ and $G(x;\phi)$ denote the probability density function (pdf) and cumulative distribution function (cdf) of a baseline model with parameter vector ϕ. Al-Shomrani et al. (2016) introduced the Topp Leone-G (TL-G) family of distributions with cdf and pdf given by

$$F_{TL-G}(x;\phi) = \left[G(x;\phi)\right]^{\lambda} \left[2 - G(x;\phi)\right]^{\lambda}, \quad \lambda > 0, \quad x \in R, \tag{1}$$

$$f_{TL-G}(x;\phi) = 2\lambda\, g(x;\phi)\bar{G}(x;\phi) \left[G(x;\phi)\right]^{\lambda-1} \left[2 - G(x;\phi)\right]^{\lambda-1}, \quad x \in R, \tag{2}$$

where, $\bar{G}(x;\phi) = 1 - G(x;\phi)$.

Furthermore, Cordeiro et al. (2013) introduced the exponentiated generalized-G (EG-G) family of distributions with cdf and pdf given below

$$F_{EG-G}(x;\phi) = \left\{1 - [1 - G(x;\phi)]^{a}\right\}^{b}, \quad , a, b > 0, x \in R, \tag{3}$$

$$f_{EG-G}(x;\phi) = a\,b\,g(x;\phi)[1 - G(x;\phi)]^{a-1}\left\{1 - [1 - G(x;\phi)]^{a}\right\}^{b-1}, \quad x \in R, \tag{4}$$

where, $a > 0$ andare two additional shape parameters. $b > 0$

The goal of this paper is to propose a new family of continuous distributions called the exponentiated generalized Topp Leone-G family (EGTL-G for short) in the gensis of TL-G and EG-G families of distributions. Some statistical properties of the new family are studied. The parameters of the proposed family are estimated via the method of maximum likelihood. Two real data sets are used to show the effectiveness of the new family.

The rest of this paper is as follows. In Section 2, we define the EGTL-G and obtain some associated reliability functions. In Section 3, the asymptotic of the EGTL-G are investigated. The expansion of EGTL -G is discussed in Section 4. In Section 5, some special models corresponding to EGTL-G are introduced. In Section 6, some statistical properties of the EGTL-G are discussed. Characterizations for the new family are presented in Section 7. In Section 8, the maximum likelihood estimates and the observed information matrix are obtained for the parameters of EGTL-G. A simulation study is conducted in Section 9. In Section 10, two applications for EGTL-G are presented. Some concluding remarks are given in the last Section.

2. The Exponentiated Generalized Topp Leone-G Family

In this section, we define the exponentiated generalized Topp leone-G family of distributions and discuss some of the reliability functions.

The cdf of the EGTL-G family can be obtained by using $F_{TL-G}(x;\phi)$ and $f_{TL-G}(x;\phi)$ given in (1) and (2) as baseline cdf and pdf in (3) as

$$F_{EGTL-G}(x)=\left\{1-\left[1-F_{TL}(x;\phi)\right]^{a}\right\}^{b}$$

$$=\left\{1-\left[1-\left(1-\bar{G}(x;\phi)^{2}\right)^{\lambda}\right]^{a}\right\}^{b},\quad a,b,\lambda>0,x\in R, \tag{5}$$

Hence forward $G(x)=G(x;\phi),\ \bar{G}(x)=1-G(x;\phi),\ g(x)=g(x;\phi).$ The pdf corresponding to (5) is

$$f_{EGTL-G}(x)=2ab\lambda\, g(x)\bar{G}(x)\left[1-\bar{G}(x)^{2}\right]^{\lambda-1}\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a-1}$$

$$\times\left\{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}\right\}^{b-1},\quad x\in R, \tag{6}$$

The EGTL-G has the following sub-families:
*If $\lambda=1$, then the EGTL-G class reduces to the EG-G family.
*If $a=b=1$, then we have the TL-G family.
The reliability function $R(x)$, hazard function $h(x)$, inverse hazard function $\tau(x)$ and cumulative hazard function $H(x)$ for the EGTL-G family are given, respectively, by

$$R(x)=1-\left\{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}\right\}^{b},\quad x\in R, \tag{7}$$

$$h(x)=\frac{2ab\lambda g(x)\bar{G}(x)\left[1-\bar{G}(x)^{2}\right]^{\lambda-1}\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a-1}\left\{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}\right\}^{b}}, \tag{8}$$

$$\tau(x)=\frac{2ab\lambda g(x)\bar{G}(x)\left[1-\bar{G}(x)^{2}\right]^{\lambda-1}\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a-1}}{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}}, \tag{9}$$

and

$$H(x)=-\ln\left(1-\left\{1-\left[1-\left(1-\bar{G}(x)^{2}\right)^{\lambda}\right]^{a}\right\}^{b}\right),\quad x\in R. \tag{10}$$

If the random variable X has pdf (6), then quantile function (qf) of X, say $Q(\mu)=F^{-1}(\mu)$, can be obtained by inverting (5). Let $G^{-1}(.)=Q_{G}(.)$ denote the qf of G, then if $U\sim(0,1)$, then

$$X_{u}=G^{-1}\left\{1-\sqrt{1-\left[1-\left(1-\mu^{1/b}\right)^{1/a}\right]^{1/\lambda}}\right\}. \tag{11}$$

3. Asymptotics

The asymptotics of cdf, pdf and hrf of EGTL-G as $x\to-\infty$ are given by

$$F(x)\sim a^{b}2^{\lambda b}G(x)^{\lambda b}\ \text{as}\ x\to-\infty;$$

$$f(x)\sim b\lambda a^{b}2^{\lambda b}g(x)G(x)^{\lambda b-1}\ \text{as}\ x\to-\infty;$$

$$h(x) \sim b\lambda\, a^{b} 2^{\lambda b} g(x)G(x)^{\lambda b-1} \text{ as } x \to -\infty .$$

The asymptotics of equations cfd, pdf and hrf of EGTL-G as $x \to \infty$ are given by

$$1-F(x) \sim b\lambda^{a}\bar{G}(x)^{2a} \text{ as } x \to \infty;$$

$$f(x) \sim 2ab\lambda^{a} g(x)\bar{G}(x)^{2a-1} \text{ as } x \to \infty;$$

$$h(x) \sim \frac{2ag(x)}{\bar{G}(x)} \text{ as } x \to \infty .$$

These results show the effect of the parameters on the tails of EGTL-G.

4. Expansion for Density and Distribution Functions

We can expand the EGTL-G family as mixture representation of the exponentiated-G family of distributions. For $|\mu|<1$ and b a positive real non-integer, we have the series representation

$$(1-u)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^{j}\,\Gamma(b)}{j\,!\Gamma(b-j)} u^{j} \tag{12}$$

Therefore using (12) in (5), the cdf of EGTL-G family can be expressed as follows:

$$F_{EGTL-G}(x) = \sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi(j,i,h,m) G(x)^{m+1}, \tag{13}$$

where, $\psi(j,i,h,m) = \frac{(-1)^{j+i+h+m}\, 2ab\lambda\Gamma(b)\Gamma(a(j+i))\Gamma(\lambda(i+1))\Gamma(2(h+1))}{j!i!h!\Gamma(m+2)\Gamma(b-j)\Gamma(a(j+i)-i)\Gamma(\lambda(i+1)-h)\Gamma(2(h+1)-m)}$.

Likewise, we can express the pdf of EGTL-G family using (11) in (6)

$$f_{EGTL-G}(x) = \sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi(j,i,h,m) S_{h+1}(x) \tag{14}$$

where, $S_{m+1}(x) = (m+1)g(x)G(x)^{m}$ is the exponentiated-G distribution with power parameter $m+1$.

5. The EGTL-G Sub-Models

In this section, we introduce three special models of the EGTL-G family.

5.1 The EGTL-Weibull (EGTLW) Model

Suppose the cdf and pdf of the Weibull distribution are the following $G(x) = 1-e^{-(\alpha x)^{\beta}}$ and $g(x) = \beta\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}}, x>0,\ \alpha,\beta>0$. respectively. Then, the cdf and pdf of EGTL-Weibull (EGTLW) distribution are, respectively, given by

$$f(x) = 2ab\,\lambda\,\beta\,\alpha^{\beta} x^{\beta-1} e^{-2(\alpha x)^{\beta}} \left[1-e^{-2(\alpha x)^{\beta}}\right]^{\lambda-1} \left\{1-\left[1-e^{-2(\alpha x)^{\beta}}\right]^{\lambda}\right\}^{a-1}$$
$$\times\left\{1-\left[1-\left(1-e^{-2(\alpha x)^{\beta}}\right)^{\lambda}\right]^{a}\right\}^{b-1}, \quad x>0,$$

and

$$F(x)=\left\{1-\left[1-\left(1-e^{-2(\alpha x)^{\beta}}\right)^{\lambda}\right]^{a}\right\}^{b}, \quad x\geq 0.$$

For $\beta=1$, the EGTLW distribution is reduced to the EGTL-exponential (EGTLE) distribution. For $\lambda=1$, we have the EG-Weibull (EGW) distribution. Moreover, $a=b=1$, then we obtain the TL-Weibull (TLW) distribution. For $a=b=\lambda=1$, the EGTLW distribution is reduced to the Exp-Weibull (EW) distribution. The plots of the density and hazard functions are displayed in Figure 1. The density introduces left, right skewed and symmetrical and reversed J shapes. In the other hand, the shape of the hazard function is increasing, decreasing, constant, bathtub and upside down bathtub.

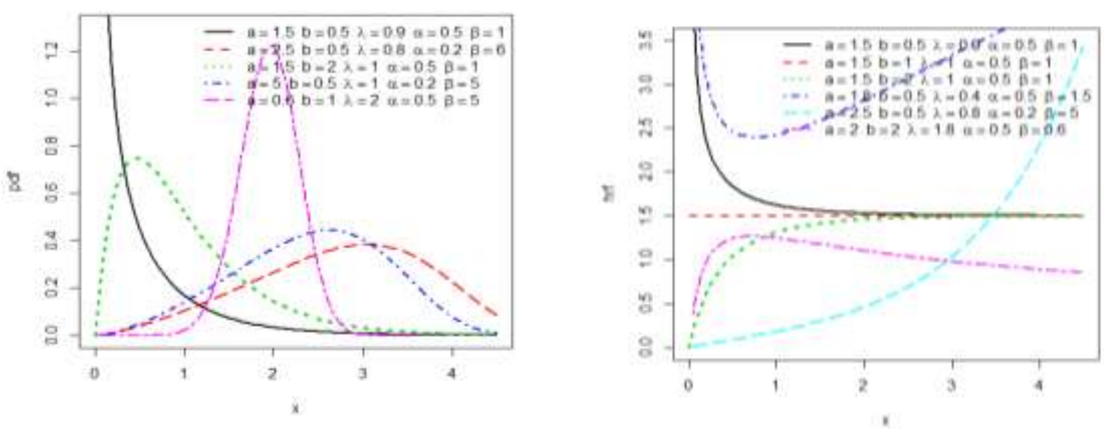

Figure 1: Plots of the EGTLW pdf and hrf for selected values of parameters.

5.2 The EGTL-Lomax (EGTLLx) Model

Consider the cdf and pdf of the lomax distribution $G(x)=1-(1+\beta x)^{-\theta}$ and $g(x)=\theta\beta(1+\beta x)^{-(\theta+1)}, x>0,\ \theta,\beta>0$ respectively. Then, the cdf and pdf of EGTL-lomax (EGTLLx) are, respectively, given by

$$f(x)=2ab\lambda\theta\beta(1+\beta x)^{-(2\theta+1)}\left[1-(1+\beta x)^{-2\theta}\right]^{\lambda-1}\left\{1-\left[1-(1+\beta x)^{-2\theta}\right]^{\lambda}\right\}^{a-1}$$
$$\times\left(1-\left\{1-\left[1-(1+\beta x)^{-2\theta}\right]^{\lambda}\right\}^{a}\right)^{b-1}, \quad x>0,$$

and

$$F(x)=\left(1-\left\{1-\left[1-(1+\beta x)^{-2\theta}\right]^{\lambda}\right\}^{a}\right)^{b}, \quad x\geq 0.$$

For $\lambda=1$, the EGTLLx distribution become the EG-lomax (EGLx) distribution. Moreover, For $a=b=1$, then the EGTLLx distribution is reduced to the TL-lomax (TLLx) distribution. For $a=b=\lambda=1$, the EGTLLx distribution is reduced to the Exp-Lomax (ELx) distribution. The plots of the density and hazard functions are given in Figure 2. The shape of the density is skeweed, reversed J shapes and near symmetric, while the hazard function introduces increasing, decreasing, and upside down bathtub shapes.

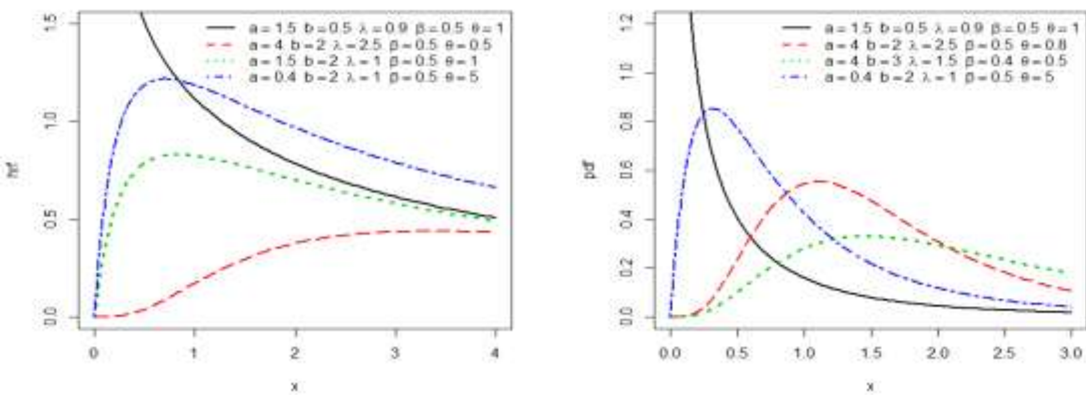

Figure 2: Plots of the EGTLLx pdf and hrf for selected values of parameters.

5.3 The EGTL-Quasi Lindley (EGTLQL) Model

The cdf and pdf of the quasi-Lindley distribution are $G(x)=1-\left(1+\frac{\theta x}{p+1}\right)e^{-\theta x}$ and $g(x)=\frac{\theta\left(p+\theta x\right)}{p+1}e^{-\theta x}, x>0,\ \ \theta>0, p>-1,$ respectively. Then, the cdf and pdf of EGTL-quasi-Lindley (EGTLQL) are, respectively, given by

$$f(x)=\left(\frac{2ab\lambda\theta(p+\theta x)}{p+1}\right)\left(1+\frac{\theta x}{p+1}\right)e^{-2\theta x}\left[1-\left(1+\frac{\theta x}{p+1}\right)^{2}e^{-2\theta x}\right]^{\lambda-1}$$
$$\times\left\{1-\left[1-\left(1+\frac{\theta x}{p+1}\right)^{2}e^{-2\theta x}\right]^{\lambda}\right\}^{a-1}\left(1-\left\{1-\left[1-\left(1+\frac{\theta x}{p+1}\right)^{2}e^{-2\theta x}\right]^{\lambda}\right\}^{a}\right)^{b-1},\quad x>0,$$

and

$$F(x)=\left(1-\left\{1-\left[1-\left(1+\frac{\theta x}{p+1}\right)^{2}e^{-2\theta x}\right]^{\lambda}\right\}^{a}\right)^{b},\ \ x\geq 0.$$

For $p=\theta$, the EGTLQL distribution is reduced to the EGTL-Lindley (EGTLL) distribution. For $\lambda=1$, we get the EG-quasi-Lindley (EGQL) distribution. Moreover, for $a=b=1$, the EGTLQL distribution becomes the TL-quasi-Lindley (TLQL) distribution. The plots of the density and hazard function are given in Figure 4. The shape of density is approximately left skewed, reversed J shapes, right skewed and symmetric, while the hazard function is increasing, decreasing, constant and bathtub shapes.

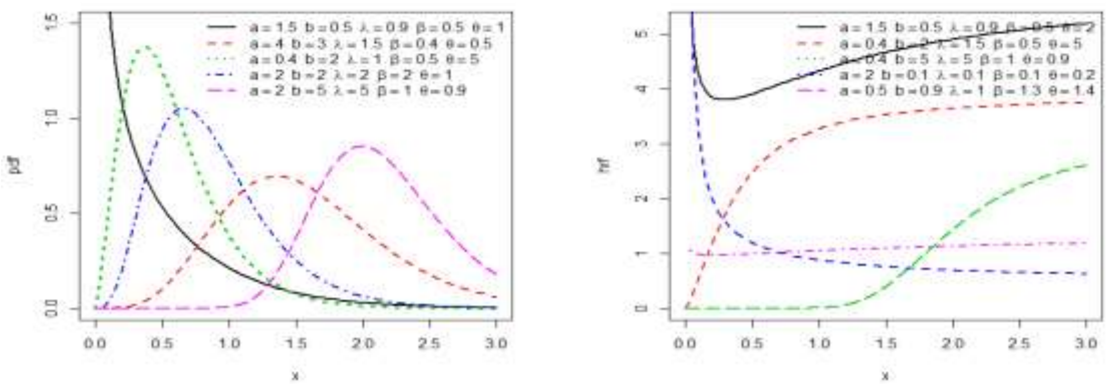

Figure 3: Plots of the EGTLQL pdf and hrf for selected values of parameters.

6. Statistical Properties

In this section, we study some statistical properties of the EGTL-G family such as: ordinary and incomplete moments, generating function, Lorenz and Bonferroni curves, Rényi entropy, stress strength model, moment of residual and reversed residual life, order statistics and extreme values.

6.1. Moments and Generating Functions

Suppose X is a random variable with EGTL-G distribution, then the ordinary moments, say μ_r', is given by

$$\begin{aligned}\mu_r' = E(X^r) &= \int_{-\infty}^{\infty} x^r \, f_{EGTL-G}(x)\, dx \\ &= \sum_{j,i,h=0}^{\infty}\sum_{m=0}^{2h+1} \psi(j,i,h,m) \int_{-\infty}^{\infty} x^r \, S_{m+1}(x)\, dx \\ &= \sum_{j,i,h=0}^{\infty}\sum_{m=0}^{2h+1} \psi^*(j,i,h,m)\, \delta_{r,m}, \qquad (15)\end{aligned}$$

where, $\psi^*(j,i,h,m) = (m+1)\psi(j,i,h,m)$ and $\delta_{r,m} = \int_{-\infty}^{\infty} x^r \, g(x) G(x)^m \, dx$ is the probability weighted moment of the baseline distribution. For integer values of n and $\mu = \mu_1' = E(X)$, one can find the nth central moment of the EGTL-G distribution, say μ_n, to be

$$\begin{aligned}\mu_n &= E\left(X - \mu_1'\right)^n \\ &= \sum_{r=0}^{n}\sum_{j,i,h=0}^{\infty}\sum_{m=0}^{2h+1} \frac{(-1)^{n-r}\,\Gamma(n+1)\left(\mu_1'\right)^{n-r}}{r!\,\Gamma(n-r+1)} \psi^*(j,i,h,m)\, \delta_{r,m}, \qquad (16)\end{aligned}$$

From (16), the measures of skewness and kurtosis of the EGTL-G distribution can be obtained as

$$\text{Skewness}(X) = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{\left(\mu_2' - \mu_1'^2\right)^{3/2}}, \tag{17}$$

and

$$\text{Kurtosis}(X) = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_3' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}, \tag{18}$$

respectively. Figure (1) shows the behavior of skewness and kurtosis of EGTL-Lx distribution.

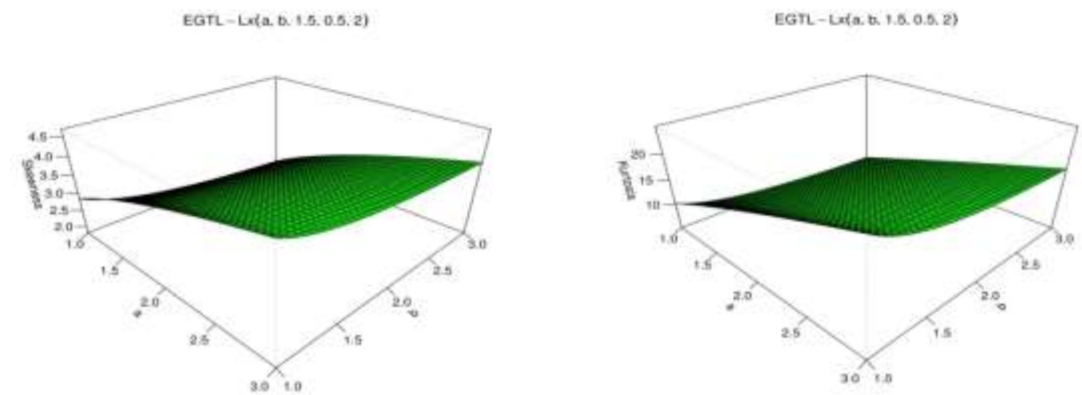

Figure 4: The skewness and kurtosis of EGTLLx distribution.

The moment and probability generating functions, denoted as $M_x(t)$, and $M_{[x]}(t)$ respectively of the EGTL-G family can be obtained based on (16) to be

$$M_x(t) = E(e^{tx}) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{2h+1}\frac{t^r}{r!}\,\psi^*(j,i,h,m)\,\delta_{r,m}. \tag{19}$$

and

$$M_{[x]}(t) = E(t^x) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{h=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{2h+1}\frac{(\ln t)^r}{r!}\,\psi^*(j,i,h,m)\,\delta_{r,m}. \tag{20}$$

6.2. Incomplete Moments

Suppose X is a random variable with EGTL-G distribution, then the rth incomplete moment, denoted by $m_r(w)$, is

$$\begin{aligned} m_r(w) &= \int_{-\infty}^{w} x^r f_{EGTL-G}(x)\,dx \\ &= \sum_{j,i,h=0}^{\infty}\sum_{m=0}^{2h+1}\psi^*(j,i,h,m)\,\tau_{r,m} \end{aligned} \tag{21}$$

where, $\tau_{r,m} = \int_{-\infty}^{w} x^r g(x) G(x)^m\,dx.$

6.3. Lorenz and Bonferroni Curves

The Lorenz and Bonferroni curves have been used in different areas such as economics, reliability, demography, insurance and medicine. The Lorenz $L_F(x)$ and Bonferroni $B(F(x))$ curves are defined respectively as follows:

$$L_F(x) = \frac{1}{E(x)} \int_0^x t\, f(t)\, dt, \qquad B(F(x)) = \frac{1}{F(x)E(x)} \int_0^x t\, f(t)\, dt = \frac{L_F(x)}{F(x)}.$$

Therefore, these quantities for the EGTL-G family are obtained from

$$L_F(x) = \frac{\sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi^*(j,i,h,m)\, \delta_{1,m}}{\sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi^*(j,i,h,m)\, \tau_{1,m}}, \tag{22}$$

and

$$B(F(x)) = \frac{\sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi^*(j,i,h,m)\, \delta_{1,m}}{F_{EGTL-G}(x) \sum_{j,i,h=0}^{\infty} \sum_{m=0}^{2h+1} \psi^*(j,i,h,m)\, \tau_{1,m}}. \tag{23}$$

6.4. Rényi Entropy

The concept of entropy has been applied in different fields such as statistics, queuing theory and reliability estimation. The Rényi entropy is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\left[\log I(\gamma)\right], \text{ where } I(\gamma) = \int f(x)^{\gamma}\, dx,\ \gamma > 0 \text{ and } \gamma \neq 0.$$

From (6), we have

$$I(\gamma) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \varpi_{j,i,\ell,m} \int_{-\infty}^{\infty} g(x)^{\gamma} G(x)^{m}\, dx,$$

where, $\varpi_{j,i,\ell,m} = (2ab\lambda)^{\gamma}(-1)^{j+i+\ell+m} \binom{\gamma(b-1)}{j} \binom{a(\gamma+j)-\gamma}{i} \binom{\lambda(\gamma+i)-\gamma}{\ell} \binom{\gamma+2\ell}{m}.$

Consequently, the Rényi entropy for the EGTL-G family is given by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\left\{ \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \varpi_{j,i,\ell,m} \int_{-\infty}^{\infty} g(x)^{\gamma} G(x)^{m}\, dx \right\}. \tag{24}$$

6.5. Stress Strength Model

The stress strength model is a common criterion used in different applications in engineering and physics. Let X_1 and X_2 be two independent random variables with EGTL-G $(a_1, b_1, \lambda_1, \phi)$ and EGTL-G $(a_2, b_2, \lambda_2, \phi)$ distributions. Then, the stress strength model is given by

$$R = \Pr(X_2 < X_1) = \int_0^{\infty} f_1(a_1, b_1, \lambda_1; \phi)\, F_2(a_2, b_2, \lambda_2; \phi)\, dx$$

$$= \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{s=0}^{\infty} \sum_{w=0}^{\infty} \sum_{h=0}^{\infty} \Omega_{j,\ell,s,w,h}, \tag{25}$$

where, $\Omega_{j,\ell,s,w,h}=\frac{(-1)^{j+\ell+s+w+h}a_1\lambda_1(h+1)^{-1}\Gamma(b_2+1)\Gamma(b_1+1)\Gamma(a_1(\ell+1))\Gamma(a_2j+1)\Gamma(\lambda_1(s+1)+\lambda_2w)}{j!\ell!s!w!h!\Gamma(b_2-j+1)\Gamma(b_1-\ell)\Gamma(a_1(\ell+1)-s)\Gamma(a_2j-w+1)\Gamma(\lambda_1(s+1)+\lambda_2w-h)}$.

6.6. Moment of Residual and Reversed Residual Life

The moment of residual and reversed residual life uniquely determine $F(x)$. The *n*th moment of the residual life, say $m_n(t)$, of a random variable X is

$$m_n(t)=E\left[(X-t)^n \middle| X>t\right]=\frac{1}{\bar{F}(t)}\int_t^{\infty}(x-t)^n\,dF(x),\ n=1,2,\dots .$$

Consequently, $m_n(t)$ for the EGTL-G family is given by

$$m_n(t)=\sum_{j,i,h,\ell=0}^{\infty}\sum_{w=0}^{n}\frac{(-1)^{n-w}\,t^{n-w}\,\Gamma(n+1)\,\psi(j,i,h,m)}{w!\,\Gamma(n-w+1)\,\bar{F}(t)}\int_t^{\infty}x^wG(x)^h\,dx. \tag{26}$$

The *n*th moment of the reversed residual life, say $M_n(t)$, of a random variable X is

$$M_n(t)=E\left[(t-X)^n \middle| X\le t\right]=\frac{1}{F(t)}\int_0^{t}(t-x)^n\,dF(x),\ n=1,2,\dots .$$

Subsequently, $M_n(t)$ for the EGTL-G family is

$$M_n(t)=\sum_{j,i,h,\ell=0}^{\infty}\sum_{w=0}^{n}\frac{(-1)^{n-w}\,t^{n-w}\,\Gamma(n+1)\,\psi(j,i,h,m)}{w!\,\Gamma(n-w+1)\,F(t)}\int_0^{t}x^wG(x)^h\,dx. \tag{27}$$

6.7. Order Statistics

Order statistics play an important role in probability and statistics. Let $X_{1,n}\le X_{2:n},\dots\le X_{n:n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:n}$, the kth order statistic is given by

$$f_{X_{k:n}}(x)=\frac{1}{\beta(k,n-k+1)}\sum_{w=0}^{n-k}(-1)^w\binom{n-k}{w}f(x)\left[F(x)\right]^{k+w-1}.$$

Based on (5) and (6), we arrive at

$$f_{X_{k:n}}(x)=\sum_{w=0}^{n-k}\sum_{j,i,h=0}^{\infty}\sum_{\ell=0}^{2h+1}\mu(w,j,i,h,\ell)\,S_{\ell+1}, \tag{28}$$

where, $\mu(w,j,i,h,\ell)=\frac{(-1)^{w+j+i+h+\ell}\;2ab\lambda}{\beta(k,n-k+1)(\ell+1)}\binom{n-k}{w}\binom{b(k+w)-1)}{j}\binom{a(j+1)-1}{gi}\binom{\lambda(i+1)-1}{h}\binom{2h+1}{\ell}$.

Moreover, the rth moment of kth order statistic for EGTL-G family is given by

$$E\left(x_{k:n}^r\right)=\sum_{w=0}^{n-k}\sum_{j,i,h=0}^{\infty}\sum_{\ell=0}^{2h+1}\mu^*(w,j,i,h,\ell)\,\delta_{r,\ell}, \tag{29}$$

where, $\mu^*(w,j,i,h,\ell)=(\ell+1)\mu(w,j,i,h,\ell)$.

6.8 Extreme values

If $\bar{X}=(X_1+\dots+X_n)/n$ denotes the mean of a random sample from (5), then by the usual central limit theorem $\sqrt{n}(\bar{X}-E(X))\big/\sqrt{Var(X)}$ approaches the standard normal distribution as $n\to\infty$ under suitable conditions. Sometimes one would be interested in the asymptotes of the extreme values $M_n=max(X_1,\dots,X_n)$ and $m_n=\min(X_1,\dots,X_n)$. .

First, Suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then by Leadbetter et al. (1987), there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty}\frac{1-G\left(t+x\,h(t)\right)}{1-G(t)}=\lim_{t\to\infty}\frac{\left(1-x\,h'(t)\right)g\left(t+x\,h(t)\right)}{g(t)}=e^{-x},$$

for every $x\in\mathbb{R}$. But

$$\lim_{t\to\infty}\frac{1-F\left(t+x\,h(t)\right)}{1-F(t)}=\lim_{t\to\infty}\frac{\left(1-x\,h'(t)\right)gf\left(t+x\,h(t)\right)}{f(t)}=e^{-2ax},$$

for every $x\in\mathbb{R}$. It follows from Leadbetter et al. (1987) that F belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{t\to\infty}P\left[a_n\left(M_n-b_n\le x\right)\right]=\exp[-\exp(-2ax)],$$

for some suitable norming constants $a_n>0$ and b_n. Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\gamma>0$ such that

$$\lim_{t\to\infty}\frac{1-G\left(t+x\,h(t)\right)}{1-G(t)}=\lim_{t\to\infty}\frac{\left(1-x\,h'(t)\right)g\left(t+x\,h(t)\right)}{g(t)}=x^{\gamma},$$

for every $x\in\mathbb{R}$. But

$$\lim_{t\to\infty}\frac{1-F\left(t+x\,h(t)\right)}{1-F(t)}=\lim_{t\to\infty}\frac{\left(1-x\,h'(t)\right)gf\left(t+x\,h(t)\right)}{f(t)}=x^{2a\gamma},$$

for every $x>0$. So, it follows from Leadbetter et al. (1987) that F belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{t\to\infty}P\left[a_n\left(M_n-b_n\le x\right)\right]=\exp[-x^{2a\gamma}],$$

for some suitable norming constants $a_n>0$ and b_n. Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then by Leadbetter et al. (1987), there must exist a $c>0$ such that

$$\lim_{t\to\infty}\frac{1-G(tx)}{G(t)}=\lim_{t\to\infty}\frac{xg(tx)}{g(t)}=x^{c},$$

for every $x>0$. But

$$\lim_{t\to\infty}\frac{F(tx)}{F(t)}=\lim_{t\to\infty}\frac{x\,F(t\,x)}{f(t)}=x^{b\lambda c},$$

for every $x>0$. Similarly it follows that F belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{t\to\infty}P\left[c_n\left(M_n-d_n\le x\right)\right]=\exp[-(-x)^{2a\gamma}],$$

for some suitable norming constants $c_n>0$ and d_n. We conclude that F belongs to the same min domain of attraction as that of G as the same argument applies to min domain of attraction. That is, F belongs to the same min domain of attraction as that of G.

7. Characterizations

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This section deals with various characterizations of the EGTL-G distribution. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function; (*iii*) the reverse (or reversed) hazard function and (*iv*) conditional expectation of a function of the random variable. It should be mentioned that for characterization (*i*) the cdf is not required to have a closed form. We present our characterizations (*i*)-(*iv*)in four subsections.

7.1 Characterizations based on two truncated moments

In this subsection we present characterizations of the EGTL-G distribution in terms of a simple relationship between two truncated moments. The first characterization result employs a theorem due to Glanzel (1987), see Theorem 1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As Shown in Glanzel (1990), this characterization is stable in the sense of weak convergence.

Theorem 1.
Let (Ω, F, P) be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ ($d=-\infty, e=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E\left[q_2(X)\middle| X\ge x\right]=E\left[q_1(X)\middle| X\ge x\right]\xi(x),\quad x\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^{-1}(H)$, $\xi\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ, particularly

$$F(x)=\int_a^x C\left|\frac{\xi'(u)}{\xi(u)q_1(u)-q_2(u)}\right|\exp\left(-s(u)\right)du,$$

where the function s is a solution of the differential equation $s'=\frac{\xi' q_1}{\xi\, q_1-q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

Proposition 7.1.
Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let

$q_1=\left\{1-\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a}\right\}^{1-b}$ and $q_2(x)=q_1(x)\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]$ for $x\in\mathbb{R}$. The random variable X has pdf (6) if and only if the function ξ defined in Theorem 1 has the form

$$\xi(x)=\frac{b}{b+1}\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right],\ x\in\mathbb{R}.$$

Proof.
Let X be a random variable with pdf (6), then

$$\left(1-F(x)\right)E\left[q_1(X)\middle|X\geq x\right]=b\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a},\quad x\in\mathbb{R},$$

and

$$\left(1-F(x)\right)E\left[q_2(X)\middle|X\geq x\right]=\frac{b}{b+1}\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a+1},\quad x\in\mathbb{R},$$

and finally

$$\xi(x)q_1(x)-q_2(x)=-\frac{1}{b+1}q_1(x)\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]<0\ \text{ for }\ x\in\mathbb{R}.$$

Conversely, if ξ is given as above, then

$$s'(x)=\frac{\xi'(x)q_1(x)}{\xi(x)\,q_1(x)-q_2(x)}=\frac{2b\lambda\,g(x)\bar{G}(x)\left[1-\bar{G}(x)^2\right]^{\lambda-1}}{1-\left(1-\bar{G}(x)^2\right)^{\lambda}}\quad x\in\mathbb{R},$$

and hence

$$s(x)=\ln\left\{\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{-b}\right\},\ x\in\mathbb{R}.$$

Now, in view of Theorem 1, X has density (6).

Corollary 7.1.

Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 7.1 the pdf of X is (6) if and only if there exist functions $q_2(x)$ and ξ defined in Theorem 1. satisfying the differential equation

$$\frac{\xi'(x)q_1(x)}{\xi(x)\,q_1(x)-q_2(x)}=\frac{2b\lambda\,g(x)\bar{G}(x)\left[1-\bar{G}(x)^2\right]^{\lambda-1}}{1-\left(1-\bar{G}(x)^2\right)^{\lambda}}\quad x\in\mathbb{R}.$$

The general solution of the differential equation in Corollary 7.1 is

$$\xi(x)=\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{-1}\left[-\int 2b\lambda\,g(x)\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}q_1(x)^{-1}q_2(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 7.1 with $D=0$. However, it should be also noted that there are triplets (q_1,q_2,ξ) satisfying the conditions of Theorem 1.

7.2 Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following proposition establishes a characterization of the EGTL-G distribution, for $b=1$, in terms of the hazard function, which is not of the above trivial form.

Proposition 7.2.
Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. For $b=1$, the pdf of X is (6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x)-\frac{g'(x)}{g(x)}h_F(x)=2a\lambda\, g(x)\frac{d}{dx}\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}}{1-\left(1-\bar{G}(x)^2\right)^{\lambda}}\right\},$$

with the boundary condition $\lim_{x\to\infty} h_F(x)=0.$

Proof. If X has pdf (6), for $b=1$, then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{q_1(x)^{-1}h_F(x)\right\}=2a\lambda\frac{d}{dx}\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}}{1-\left(1-\bar{G}(x)^2\right)^{\lambda}}\right\},$$

or

$$h_F(x)=2a\lambda g(x)\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}}{1-\left(1-\bar{G}(x)^2\right)^{\lambda}}\right\},$$

which is the hazard function of the EGTL-G distribution for $b=1.$

7.3 Characterization in terms of the reverse (or reversed) hazard function

The reverse hazard function, τ_F, of a twice differentiable distribution function, F, is defined as

$$\tau_F=\frac{f(x)}{F(x)},\quad x\in\text{support of } F.$$

Proposition 7.3
Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The pdf of X is (6) if and only if its reverse hazard function τ_F satisfies the differential equation

$$\tau_F'(x)-\frac{g'(x)}{g(x)}\tau_F(x)=2a\lambda\, g(x)\frac{d}{dx}\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a-1}}{1-\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a}}\right\}.$$

Proof.
If X has pdf (6), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\left(g(x)\right)^{-1}\tau_F(x)\right\}=2ab\lambda\frac{d}{dx}\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a-1}}{1-\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a}}\right\},$$

or

$$\tau_F(x) = 2ab\lambda g(x)\left\{\frac{\bar{G}(x)\left(1-\bar{G}(x)^2\right)^{\lambda-1}\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a-1}}{1-\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a}}\right\},$$

which is the reverse hazard function of the EGTL-G distribution.

7.4 Characterizations Based on Conditional Expectation

The following propositions have already appeared in Hamedani (2013), so we will just state them here which can be used to characterize the EGTL-G distribution.

Proposition 7.4.1

Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E\left[\psi(X)\middle| X \geq x\right] = \delta\psi(x), \quad x \in (a,b),$$

if and only if

$$\psi(x) = \left(1-F(x)\right)^{\frac{1}{\delta}-1,} \qquad x \in (a,b).$$

Proposition 7.4.2

Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on (a,b) with $\lim_{x\to b^+} \psi_1(x) = 1$. Then for $\delta_1 \neq 1$,

$$E\left[\psi_1(X)\middle| X \leq x\right] = \delta_1\psi_1(x), \quad x \in (a,b),$$

implies

$$\psi_1(x) = \left(F(x)\right)^{\frac{1}{\delta_1}-1,} \qquad x \in (a,b).$$

Remarks 7.4

(a) For $\psi(x) = 1-\left(1-\bar{G}^2(x)\right)^{\lambda}$, $b=1$, $\delta = \frac{a}{a+1}$ and $(a,b) = \mathbb{R}$, Proposition 7.4.1 provides a characterization of the EGTL-G distribution. (b) For $\psi_1(X) = 1-\left[1-\left(1-\bar{G}(x)^2\right)^{\lambda}\right]^{a}$, $\delta_1 = \frac{b}{b+1}$ and $(a,b) = \mathbb{R}$, Proposition 6.4.2 provides a characterization of the EGTL-G distribution. (c) Of course there are other suitable functions than the ones we mentioned above, which are chosen for simplicity.

8. Estimation of Parameters

In this section, we describe the maximum likelihood estimates (MLEs) for the model parameters of the EGTL-G family. Letbe an independent random sample $x_1, x_2, ..., x_n$ from EGTL-G family with set of parameters $\Theta = (a,b,\lambda,\phi)^T$, then the corresponding log-likelihood function is given by

$$\ell = n\left[\log(2)+\log(a)+\log(b)+\log(\lambda)\right]+\sum_{i=1}^{n}\log\left(g(x_i,\phi)\right)+\sum_{i=1}^{n}\log\left(\bar{G}(x_i,\phi)\right)$$
$$+(\lambda-1)\sum_{i=1}^{n}\log(\varepsilon_i)+(a-1)\log\left(1-\varepsilon_i^{\lambda}\right)+(b-1)\sum_{i=1}^{n}\log\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right], \tag{30}$$

where, $\varepsilon_i = 1-\bar{G}(x_i,\phi)^2$.

The components of the score vector $\nabla\ell = \left(\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\phi}\right)$are

$$\frac{\partial\ell}{\partial a} = \frac{n}{a}+\sum_{i=1}^{n}\log\left(1-\varepsilon_i^{\lambda}\right)-(b-1)\sum_{i=1}^{n}\left\{\frac{\left(1-\varepsilon_i^{\lambda}\right)^a\log(1-\varepsilon_i^{\lambda})}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right\}, \tag{31}$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b}+\sum_{i=1}^{n}\log\left\{1-(1-\varepsilon_i^{\lambda})^a\right\}, \tag{32}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda}+\sum_{i=1}^{n}\log\left(\varepsilon_i\right)-(a-1)\sum_{i=1}^{n}\left[\frac{\varepsilon_i^{\lambda}\log\left(\varepsilon_i\right)}{1-\varepsilon_i^{\lambda}}\right]$$
$$+a(b-1)\sum_{i=1}^{n}\left[\frac{\varepsilon_i^{\lambda}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}\log\left(\varepsilon_i\right)}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right], \tag{33}$$

and (for $r=1,...,q$)

$$\frac{\partial\ell}{\partial\phi_r} = \sum_{i=1}^{n}\left\{\frac{g_r'(x_i,\phi)}{g_r(x_i,\phi)}\right\}-\sum_{i=1}^{n}\left\{\frac{G_r'(x_i,\phi)}{\bar{G}_r(x_i,\phi)}\right\}+2(\lambda-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)\,G_r'(x_i,\phi)}{\varepsilon_i}\right\}$$
$$-2\lambda(a-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)\,G_r'(x_i,\phi)\varepsilon_i^{\lambda-1}}{1-\varepsilon_i^{\lambda}}\right\}$$
$$+2a\lambda(b-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)G_r'(x_i,\phi)\varepsilon_i^{\lambda-1}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right\}, \tag{34}$$

where, $g_r'(x_i,\phi) = \partial g_r(x_i,\phi)/\partial\phi_r$ and $G_r'(x_i,\phi) = \partial G_r(x_i,\phi)/\partial\phi_r$.

The MLEs, say $\hat{\Theta} = (\hat{a},\hat{b},\hat{\lambda},\hat{\phi})$ of $\Theta = (a,b,\lambda,\phi)^T$ can be obtained by solving the system of nonlinear equations (31) through (34). These equations cannot be solved analytically and it needed iterative techniques such as Newton-Raphson algorithm.

For the purposes of interval estimation and testing hypotheses for the vector parameters $\Theta = (a,b,\lambda,\phi)^T$, we derive the $(q+3)\times(q+3)$ observed information matrix $J(\Theta) = \left\{J_{wv}\right\}$ (for $w,v = a,b,\lambda,\phi_r$) to be

$$J(\Theta)=\begin{bmatrix} J_{aa} & J_{ab} & J_{a\lambda} & J_{a\phi_r} \\ J_{ba} & J_{bb} & J_{b\lambda} & J_{b\phi_r} \\ J_{\lambda a} & J_{\lambda b} & J_{\lambda\lambda} & J_{\lambda\phi_r} \\ J_{\phi_r a} & J_{\phi_r b} & J_{\phi_r\lambda} & J_{\phi_r\phi_r} \end{bmatrix}$$

whose elements are given in Appendix A.

9. Simulation Study

In this section, the maximum likelihood estimators for the parameters of EGTLLx density function have been assessed by simulating: $(\beta,\theta,\lambda,a,b)=(0.5,2,1.5,2,1.5)$. To verify the validity of the maximum likelihood estimators, the bias and the mean square error of MLE have been used. For example, for $(\beta,\theta,\lambda,a,b)=(0.5,2,1.5,2,1.5)$, $r=1000$ times, samples of $n=30,80,...,280$ of EGTLLx (0.5,2,1.5,2,1.5) have been simulated. To estimate the numerical values of the maximum likelihood estimators, the *optim* function (in the *stat* package) and Nelder-Mead method in R software have been used. If $\xi=(\beta,\theta,\lambda,a,b),$ for any simulation by n volume and $i=1,2,...,r,$ the maximum likelihood estimates are obtained as $\hat{\xi}_i=(\hat{\beta}_i,\hat{\theta}_i,\hat{\lambda}_i,\hat{a}_i,\hat{b}_i).$

To examine the performance of the MLEs for the EGTLLx distribution, we perform a simulation study as follows:

1. Generate r samples of size n from pdf of EGTL-Lx.
2. Compute the MLEs for the r samples, say $(\hat{\beta}_i,\hat{\theta}_i,\hat{\lambda}_i,\hat{a}_i,\hat{b}_i)$. for $i=1,2,...,r=1000.$
3. Compute the standard errors of the MLEs for r samples, say $(s_{\hat{\beta}},s_{\hat{\theta}_i},s_{\hat{\lambda}},s_{\hat{a}_i},s_{\hat{b}_i})$ for $i=1,2,...,r.$
4. Compute the biases and mean squared errors given by

$$Bias_{\hat{\xi}}(n)=\frac{1}{r}\sum_{i=1}^{r}\left(\hat{\xi}_i-\xi_i\right),$$

and

$$MSE_{\hat{\xi}}(n)=\frac{1}{r}\sum_{i=1}^{r}\left(\hat{\xi}_i-\xi_i\right)^2,$$

for $\xi=(\beta,\theta,\lambda,a,b).$

We repeat these steps for $r=1000$ and $n=30,80,...,n^*$ (n^* is different in each issue) with different values of $(\beta,\theta,\lambda,a,b),$ so computing $\text{Bias}_{\hat{\xi}}(n),\text{MSE}_{\hat{\xi}}(n)$. The results of the empirical study were conducted in Table 1. It is observed, from Table 1, reveals how the biases, mean squared errors vary with respect to $n.$ As expected, the Biases and MSEs of the estimated parameters converge to zero while n growing.

Table 1: Biases and MSEs for the MLEs of the parameters of the EGTLLx distribution.

Groups	Initial Values	Bias and MSE	Sample sizes					
			$n=30$	$n=80$	$n=130$	$n=180$	$n=230$	$n=280$
I	$\beta=0.5$	Bias	0.588282	0.463250	0.253137	0.196429	0.196585	0.197232
		MSE	0.750044	0.379374	0.113175	0.050236	0.040844	0.039935

	$\theta=2$	Bias	0.318604	0.189122	0.050938	0.008020	0.001305	-0.000105
		MSE	0.220227	0.082685	0.017103	0.002727	0.000477	0.000021
	$\lambda=1.5$	Bias	-0.612847	-0.310104	-0.092810	-0.012416	0.001024	0.002189
		MSE	0.827126	0.231947	0.054623	0.011199	0.002129	0.000541
	$a=2$	Bias	0.281671	0.162439	0.041882	0.003197	-0.000315	-0.000269
		MSE	0.162758	0.052194	0.011112	0.001623	0.000255	0.000025
	$b=1.5$	Bias	-0.316271	-0.190701	-0.028233	0.000994	0.000403	0.000534
		MSE	0.340274	0.127963	0.023873	0.004133	0.000654	0.000139
II	$\beta=1.5$	Bias	-0.10172	-0.076118	-0.004988	0.045532	0.047782	0.033494
		MSE	1.12876	0.659925	0.464142	0.320768	0.330558	0.259729
	$\theta=2$	Bias	-0.00935	0.058297	0.002742	-0.006870	0.038627	-0.003233
		MSE	0.74986	0.407614	0.405201	0.387705	0.348475	0.355531
	$\lambda=1.5$	Bias	0.09847	0.184462	0.198048	0.222955	0.236501	0.274003
		MSE	1.30764	0.728996	0.518753	0.359886	0.352336	0.290443
	$a=2$	Bias	0.13525	0.064509	0.031263	0.039512	0.049806	0.034119
		MSE	0.51769	0.255398	0.206680	0.182507	0.166377	0.170890
	$b=1.5$	Bias	0.03310	-0.084521	-0.038048	-0.091139	-0.116460	-0.102970
		MSE	1.15560	0.832793	0.746394	0.772271	0.758404	0.758816
III	$\beta=1.5$	Bias	0.29654	0.43626	0.41452	0.38514	0.335512	0.361434
		MSE	1.20911	0.75775	0.48009	0.36574	0.301555	0.295165
	$\theta=2$	Bias	0.37086	0.39527	0.38307	0.37004	0.375806	0.389026
		MSE	0.76247	0.70747	0.52296	0.46196	0.373752	0.345372
	$\lambda=1.5$	Bias	-0.09290	-0.03078	0.03592	0.05610	0.025042	-0.004890
		MSE	1.18423	0.55924	0.27154	0.21421	0.156952	0.137428
	$a=2$	Bias	0.23905	0.31546	0.34941	0.33782	0.305165	0.312768
		MSE	0.48537	0.30378	0.25217	0.23322	0.183310	0.181520
	$b=3$	Bias	-0.45371	-0.74413	-0.82078	-0.71431	-0.622516	-0.648932
		MSE	1.61957	1.45819	1.47566	1.14898	0.885651	0.903368

10. Applications

In this section, we provide two application to real data to illustrate the applicability of the EGTL-G family. We focus on the EGTLLx distribution introduced in Subsection 6.2. We have used data from Nigm et al. (2003) and is about ordered failure of components. The data is given as follows: 0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.

The second data set consists of 63 observations of the strengths of 1.5 cm glass fibers which obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11,1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. This data recently study by Reyad and Othman (2017).

The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization (L-BFGS-B) and the log-likelihood function evaluated. The goodness-of-fit measures, Anderson-Darling (A^*), Cramér–von Mises (W^*) are computed. The lower the values of these criteria, the better the fit. The value for the Kolmogorov Smirnov (KS) statistic and its p-value are also provided.

We compare the EGTLLx distribution with those of the Lomax (Lx), beta Lomax (BLx) (Lemonte and Cordeiro, 2013), exponentiated Lomax (ELx) (El-Bassiouny et al., 2015) Kumaraswamy Lomax (KwLx) (Lemonte and Cordeiro, 2013), Topp-Leone Gamma (TLGa) (Al-Shomrani et al., 2016), McDonald Lomax (McLx) (Lemonte and Cordeiro, 2013), Topp-Leone Lomax (TLLx) (Al-Shomrani et al., 2016) and Topp-Leone exponential (TLE) (Al-Shomrani et al., 2016), The MLEs and some statistics of the models for the first data set and second data set are presented in Tables 2, 3, 4 and 5 respectively

Table 2: The MLEs for the first data set

Model	Estimates with standard error in parenthesis				
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$
EGTLLx	6.7320 (4.7607)	0.1514 (0.1278)	4.5746 (1.7120)	0.2128 (0.2473)	5.8854 (1.7751)
BLx	0.8075 (0.2211)	10.8542 (139.0261)	--- ---	26.0677 (148.8267)	12.1996 (153.1953)
TLGa	0.8603 (0.5353)	14.8296 (45.0796)	0.0108 (0.0235)	3.3653 (7.8418)	7.9758 (12.9886)
McLx	0.6596 (0.2054)	16.9164 (133.4944)	2.9236 (5.7999)	9.6612 (136.6751)	10.5497 (165.7145)
KwLx	46.4929 (127.0901)	3.9739 (12.8003)	0.7426 (0.4514)	0.4890 (1.9023)	--- ---
Elx	8.1186 (24.9598)	44.1626 (132.4419)	0.7986 (0.2249)	--- ---	--- ---
TLE	0.7925 (0.2208)	2.6644 (0.8240)	--- ---	--- ---	--- ---
TLLx	13.7262 (54.6940)	37.0005 (145.1542)	0.7958823 (0.2233255)	--- ---	--- ---
Lx	27.4680 (250.6248)	171.2326 (155.7832)	--- ---	--- ---	--- ---

Table 3: Some statistics for the models fitted to the first data set.

Model	Goodness of fit criteria					
	A*	W*	KS	P-value	AIC	BIC
EGTLLx	0.1424	0.0234	0.0916	0.9902	24.5612	19.5825
BLx	0.2206	0.0389	0.1229	0.8876	25.6264	21.6435
TLGa	0.1430	0.0247	0.0957	0.9809	24.5897	19.5911
McLx	0.1428	0.0235	0.0963	0.9852	24.5547	19.5760
KwLx	0.2111	0.0373	0.1234	0.8844	25.6777	21.6947
ELx	0.2213	0.0391	0.1241	0.8806	28.9779	26.9864
TLE	0.2124	0.0375	0.1225	0.8899	27.6581	24.6709
TLLx	0.2178	0.0385	0.1234	0.8847	27.6144	24.6272
Lx	0.2184	0.0386	0.1099	0.9065	28.9779	26.9864

Table 4: The MLEs for the second data set

Model	Estimates with standard error in parenthesis				
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$
EGTLLx	5.7322 (25.8626)	0.4767 (0.4310)	1.8787 (2.1570)	0.4327 (0.2677)	5.4623 (4.5758)
BLx	17.8478 (3.4499)	40.1913 (60.2638)	--- ---	77.5816 (222.1545)	19.3027 (41.0907)

TLGa	28.7175 (20.1487)	0.4032 (0.2492)	1.4064 (1.8410)	13.4036 (21.2440)	4.3984 (7.9034)
McLx	9.0850 (2.4677)	116.5477 (15.6533)	22.4273 (4.0311)	45.2586 (11.6442)	41.2484 (10.14361)
KwLx	27.1531 (67.0525)	46.1506 (125.0250)	9.0710 (2.3638)	91.7567 (89.5981)	--- ---
ELx	43.2618 (53.4628)	116.1829 (139.3477)	32.7701 (10.3498)	--- ---	--- ---
TLE	31.3565 (9.5283)	1.3057 (0.1190)	--- ---	--- ---	--- ---
TLLx	80.2159 (19.8572)	106.8310 (156.5348)	32.46991 (9.9803)	--- ---	--- ---
Lx	211.2298 (322.7742)	140.5679 (214.5268)	--- ---	--- ---	--- ---

Table 5: Some statistics for the models fitted to the second data set.

Model	Goodness of fit criteria					
	A*	W*	KS	P-value	AIC	BIC
EGTLLx	1.0842	0.1951	0.1453	0.1381	39.0461	49.7618
BLx	3.1526	0.5748	0.2165	0.0054	56.2913	56.9810
TLGa	1.2965	0.2353	0.1617	0.0740	40.4989	51.2146
McLx	1.4040	0.2558	0.1700	0.05238	41.0887	51.8043
KwLx	1.7966	0.3281	0.1731	0.0457	42.7185	51.2911
Elx	4.3439	0.7969	0.2285	0.0027	69.4747	75.9041
TLE	4.2870	0.7861	0.2295	0.0027	66.7669	71.0532
TLLx	4.3256	0.7935	0.23121	0.0024	69.1780	75.6074
Lx	3.1354	0.5717	0.4179	0.0000	182.0878	186.3741

The values in Tables 3 and 5 indicate that the EGTLLx model has the lowest values for A*, W*, KS and largest P-values among all fitted models (for the two real data sets). So, the EGTLLx models could be chosen as the best models. The estimated pdfs and cdfs plots are displayed in Figures (5) and (6). It is clear from Figures (5) and (6) that the new EGTLLx distribution provides the best fits to both data sets.

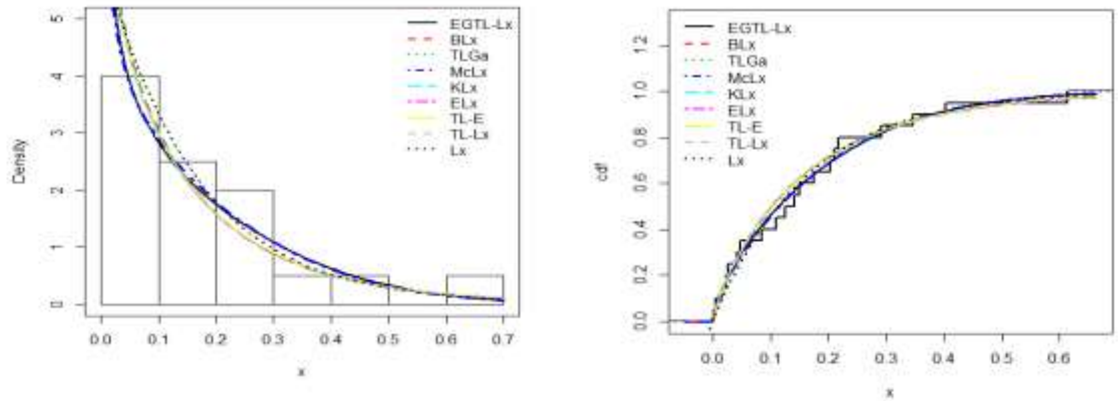

Figure 5: Estimated pdfs and cdfs plots of the EGTL-Lx distribution for data set 1

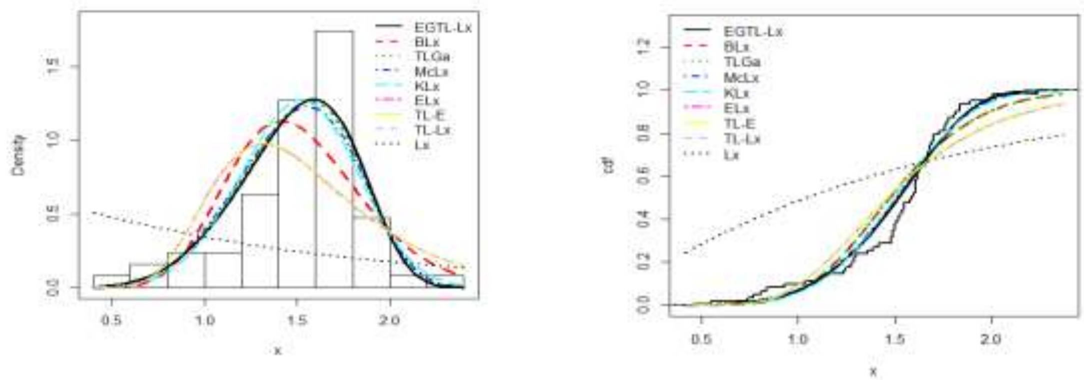

Figure 6: Estimated pdfs and cdfs plots of the EGTL-Lx distribution for data set 2

Moreover, The likelihood ratio (LR) statistic can be used to check if the EGTLLx distribution is strictly "superior" to the ELx distribution for a given data sets. The test of $H_0 = a = b = \lambda = 1$ versus $H_1 = H_0$ is not true is equivalent to compare the EGTLLx and ELx distributions, where $\hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{a}$ and $\hat{b}$ are the MLEs under H_1 and $\hat{\beta}, \hat{\theta}$ are the MLEs under H_0, is asymptotically follows chi-square distribution with 3 degrees of freedom. The LR statistics for testing the hypotheses H_0 : ELx against H_1 : EGTLLx for data sets 1 and 2 are 81.5 and 34.42, respectively, and all yield *Pvalues* < 0.00001. Thus, we can reject the null hypotheses in all cases in favor of the EGTLLx distribution at any usual significance level; that is, the EGTLLx model is significantly better than the ELx distribution.

11. Conclusions

We propose a new class of distributions, called the exponentiated generalized Topp Leone-G (EGTL-G) family by using the TL-G family as a baseline model in the EG-G class of distributions. We investigate the statistical properties of the suggested family such as ordinary and incomplete moments, generating functions, Lorenz and Bonferroni curves, Rényi of entropy, stress strength model, moment of residual and reversed residual lifes, order statistics and extreme values. The method of maximum likelihood is applied to estimate the model parameters and the observed information matrix is discussed. Two real data sets are used to show that some models corresponding to the EGTL-G family can give better fit than similar models generated by well-known families.

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Appendix A

The elements of the observed information matrix are given below

$$J_{aa} = \frac{-n}{a^2} - (b-1)\sum_{i=1}^{n}\left\{\frac{\left(1-\varepsilon_i^{\lambda}\right)^a \left(\log(1-\varepsilon_i^{\lambda})\right)^2}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

$$J_{ab} = -\sum_{i=1}^{n}\left[\frac{\left(1-\varepsilon_i^{\lambda}\right)^a \log(1-\varepsilon_i^{\lambda})}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right],$$

$$J_{a\lambda} = -\sum_{i=1}^{n}\left[\frac{\varepsilon_i^{\lambda}\log(\varepsilon_i)}{1-\varepsilon_i^{\lambda}}\right] - (b-1)\sum_{i=1}^{n}\left\{\frac{\varepsilon_i^{\lambda}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}\log(\varepsilon_i)\ \xi_i}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

$$J_{a\phi_r} = -2\lambda\sum_{i=1}^{n}\left(\frac{\bar{G}_r(x_i,\phi)\ G_r'(x_i,\phi)\ \varepsilon_i^{\lambda-1}}{1-\varepsilon_i^{\lambda}}\right) - 2\lambda(b-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)\ G_r'(x_i,\phi)\ \varepsilon_i^{\lambda-1}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}\ \kappa_i}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

$$J_{bb} = \frac{-n}{b^2},$$

$$J_{b\lambda} = a\sum_{i=1}^{n}\left(\frac{\varepsilon_i^{\lambda}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}\log(\varepsilon_i)}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right),$$

$$J_{b\phi_r} = 2\lambda a\sum_{i=1}^{n}\left(\frac{\bar{G}_r(x_i,\phi)G_r'(x_i,\phi)\varepsilon_i^{\lambda-1}\left(1-\varepsilon_i^{\lambda}\right)^{a-1}}{1-\left(1-\varepsilon_i^{\lambda}\right)^a}\right),$$

$$J_{\lambda\lambda}=\frac{-n}{\lambda^2}-(a-1)\sum_{i=1}^{n}\left[\frac{\varepsilon_i^{\lambda}\left(\ln(\varepsilon_i)\right)^2}{\left(1-\varepsilon_i^{\lambda}\right)^2}\right]+a(b-1)\sum_{i=1}^{n}\left\{\frac{\varepsilon_i^{\lambda}\left(1-\varepsilon_i^{\lambda}\right)^{a-2}\left(\ln(\varepsilon_i)\right)^2\gamma_i}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

$$J_{\lambda\phi_r}=2\sum_{i=1}^{n}\left(\frac{\bar{G}_r(x_i,\phi)\,G_r'(x_i,\phi)}{\varepsilon_i}\right)-2(a-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)\,G_r'(x_i,\phi)\varepsilon_i^{\lambda-1}\left[1-\varepsilon_i^{\lambda}+\ln(\varepsilon_i^{\lambda})\right]}{\left(1-\varepsilon_i^{\lambda}\right)^2}\right\}$$

$$+2a(b-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}_r(x_i,\phi)\,G_r'(x_i,\phi)\,\varepsilon_i^{\lambda-1}\left(1-\varepsilon_i^{\lambda}\right)^{a-2}\,\Omega_i}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

$$J_{\phi_r\phi_r}=\sum_{i=1}^{n}(q_i)-\sum_{i=1}^{n}(s_i)-2(\lambda-1)\sum_{i=1}^{n}\left(\frac{s_i-\bar{G}_r(x_i,\phi)^2}{\varepsilon_i}\right)-2\lambda(a-1)\sum_{i=1}^{n}\left(\frac{\varepsilon_i^{\lambda-2}\left(\Upsilon_i+\Lambda_i\right)}{\left(1-\varepsilon_i^{\lambda}\right)^2}\right)$$

$$-2ab(\lambda-1)\sum_{i=1}^{n}\left\{\frac{\varepsilon_i^{\lambda-2}\left(1-\varepsilon_i^{\lambda}\right)^{a-2}\left\{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]\left[\upsilon_i+\Delta_i\right]-\Phi_i\right\}}{\left[1-\left(1-\varepsilon_i^{\lambda}\right)^a\right]^2}\right\},$$

where,

$$\xi_i=\left[1-2\varepsilon_i^{\lambda}\right]^a\log\left(1-\varepsilon_i^{\lambda}\right)^a+\left[1-\varepsilon_i^{\lambda}\right]^a-1,$$

$$\kappa_i=\left[1-2\left(1-\varepsilon_i^{\lambda}\right)^a\right]\log\left(1-\varepsilon_i^{\lambda}\right)^a+\left(1-\varepsilon_i^{\lambda}\right)^a-1,$$

$$\gamma_i=1-a\varepsilon_i^{\lambda}-\left(1-\varepsilon_i^{\lambda}\right)^a,$$

$$\Omega_i=1+\lambda-\left(1-\varepsilon_i^{\lambda}\right)^a\left[1+a\varepsilon_i\log(\varepsilon_i)\right]-\varepsilon_i^{\lambda}\left[1-a\lambda-(1-2a)\varepsilon_i^{\lambda}\right],$$

$$q_i=\frac{g_r'(x_i,\phi)\,g_r''(x_i,\phi)-g_r'(x_i,\phi)^2}{g_r(x_i,\phi)^2},$$

$$s_i=\frac{\bar{G}_r(x_i,\phi)\,G_r''(x_i,\phi)+G_r'(x_i,\phi)^2}{\bar{G}_r(x_i,\phi)^2},$$

$$\Upsilon_i=\left(1-\varepsilon_i^{\lambda}\right)\left\{\bar{G}_r(x_i,\phi)^2\varepsilon_i\,G_r''(x_i,\phi)+G_r'^2(x_i,\phi)\left[(2\lambda-1)\,\bar{G}_r(x_i,\phi)^2-1\right]\right\},$$

$$\Lambda_i=2\lambda\,\bar{G}_r(x_i,\phi)^2\,\varepsilon_i^{\lambda}\,G_r'(x_i,\phi)^2,$$

$$\upsilon_i=\bar{G}_r(x_i,\phi)\,\varepsilon_i\left[G_r''(x_i,\phi)\left(1-\varepsilon_i^{\lambda}\right)-2\lambda(a-1)\,\bar{G}_r(x_i,\phi)\;\varepsilon_i^{\lambda-1}\,G_r'(x_i,\phi)^2\right],$$

$$\Delta_i=G_r'(x_i,\phi)^2\left[(2\lambda-1)\bar{G}_r(x_i,\phi)^2-1\right]\left(1-\varepsilon_i^{\lambda}\right),$$

$$\Phi_i=2\lambda a\,\bar{G}_r(x_i,\phi)^2\,\varepsilon_i^{\lambda}\,G_r'(x_i,\phi)^2\,\left(1-\varepsilon_i^{\lambda}\right)^a.$$