Impatient Customers in an M/M/c queue with Single and Multiple Synchronous Working Vacations

Shakir Majid
Department of Mathematics
Annamalai University, India
shakirku16754@gmail.com

P. Manoharan
Department of Mathematics
Annamalai University India
drmanomaths.hari@gmail.com

Abstract

In this paper, an M/M/c queuing model with synchronous working vacation and impatient customers is considered. The model is analyzed for two vacation policies i.e. multiple working vacation (MWV) policy and single working vacation (SWV) policy. The servers serve the customers at a slower rate than the normal busy period during a working vacation and this becomes the cause of customer's impatience. The M/M/c queue with two such policies is described and using the PGF method, we obtain various system performance measures in terms of two indicators. We have derived some results relating to the limiting behavior of some performance measures. At the end of each model, we have presented some numerical examples to demonstrate the effects of system parameters on some performance measures. Finally, a comparison between the two models is carried out.

Keywords: M/M/c queue; Synchronous working vacation; Impatient customers; Generating function.

1. Introduction

Impatient customers in queuing models occur in several life scenarios such as those involving impatient telephone switchboard customers, hospital emergency rooms handling critical patients and inventory systems that store perishable goods Obert (1979) and Al-Seedyet al. (2009). Many researchers are interested in analyzing the queuing models with impatient customers and considered the impatience behavior by various directions, due to potential applications of queuing systems in call centers, communication networks, production-inventory systems and several other areas Bonaldand Roberts (2001) and Benjaafaret al.(2010). The first to investigate the impatient phenomenon in queuing models appears to be Palms pioneering work Palm (1953)by considering the infinite buffer M/M/c queue where each arriving customers remains in the queue until his waiting time does not exceed the impatient time which is exponentially distributed. Daley (1965) analyzed the impatient phenomenon in $GI/G/1$ queuing system in which the customers may leave the system if their waiting time is too long before starting or completing their service. Delay obtained an integral equation for the limiting distribution function and analyzed solution for the deterministic and distributed impatience. Takacs(1974) further analyzed the M/G/1 queuing system in which customers sojourn time have a static threshold and obtained the actual and virtual limiting waiting time distributions. And these results are generalized in different direction by several authorsBaccelli et al. (1984), Boxma and de Waal (1994), Van Houdt et al.

(2003) and Yue and Yue (2009). In the above mentioned literature, the cause of impatience was either a long wait already experienced by a customer upon arrival at a queue, or a long wait anticipated by a customer upon arrival. However, Altman and Yechiali (2006) and (2007) studied queuing models with impatient customers where the cause of impatience becomes the server's vacation and unavailability of server upon arrival. Hence, the cause of the impatience is the unavailability of the server. The M/M/1, M/G/1 and M/M/c queues were analyzed in Altman and Yechiali (2006), whereas $M/M/\infty$ queue was studied in Altman and Yechiali (2007). Yechiali(2007) investigated the queuing model with system disasters where the customers turned to be impatient only when the system is down. This work was broaden and enhanced by Economou and Kapodistria(2010) who studied synchronized abandonments in queuing models. Perel and Yechiali(2010) analyzed M/M/c queuing system with impatient customers operating in a 2-phase (fast and slow) Markovian random environment. Customers became impatient because of the slow service rate when the system works in a slow phase. Yue et al. (2006) andKawanishi(2008) investigated the impatience behavior of a finite capacity multi-server queuing system.

M/M/c queuing system with vacation were first studied by Levy and Yechiali(1976) where each sever takes the vacation individually (called asynchronous vacations). Later, Chao and Zhao (1998) investigate the M/M/c model for both synchronous and asynchronous vacation policies. Zhang and Tian (2003a and 2003b) carried out the analysis of a multi-server queue with asynchronous and synchronous vacation policies of a finite number of servers. A multi-server queuing model with Markovian arrival and synchronous phase type vacations was formulated by Chakravarthy(2007) with the help of probabilistic rule and controlled thresholds.

In the above mentioned study, we have assumed that the server halts service during the vacation. However, there is lot of examples where the server does not completely stop serving the customers during the vacation. Rather, it will render service at a lower rate to the queue. Servi and Finn (2002) were the first to introduce this kind of vacation policy, called working vacation policy and studied an M/M/1/WV queuing model where service times during a non-vacation period, the service times during a working vacation, and the vacation times are all assumed to be exponentially distributed with different rates. Kim et al. (2003) and Wu and Takagi (2006) generalized thework of Servi and Finn (2002) to an M/G/1 queue with working vacation. Baba (2005) extended this study by using the matrix-geometric method to a GI/M/1 queue with working vacation. Tian, Zhao, and Wang (2008) investigated the M/M/1 queue with single working vacation. Banik et al. (2007) studied the GI/M/1/N queue with multiple working vacations and computed a series of numerical results. Jain andUpadhyaya(2011) analyzed a finite-buffer multi-server unreliable Markovian queue with synchronous working vacation policy. Banik(2010) studied the $GI/M/1/N$ and $GI/M/1/\infty$ queuing models for single working vacation. Recently, Selvaraju and Goswami(2013) analyzed the M/M/1 queue with single and multiple working vacation and impatient customers. They computed closed form solution and various performance measures with stochastic decomposition for both the working vacation policies.

In a classical vacation queuing system, the server completely stops service to the customers during the vacation and customer has to wait for the service till the regular busy period starts and the vacation period ends. However, if the server serves the customers at a lower rate during the vacation period rather than halts the service completely, we get the working vacation (WV) policy Servi and Finn (2002). Hence, there is a quite a difference between the working vacation queue and classical vacation queue. During the vacation, customers in the working vacation queue depart the system after getting served, but customers in the classical vacation queue cannot leave the system after getting served. During the vacation, the number of customers can only increase in a classical vacation; however, the number of customers in a working vacation can increase or decrease. Hence, the working vacation queuing models have more complicated modalities and the analysis of such kind of models is far more complex than the classical vacation.

The paper is organized as follows. We provide the model description of the $M/M/c$ queue with MWV in section 2. In section 3, we formulate the model as a quasi-birth-death process, the steady state differential equations are derived and their solutions are presented. We also derive various performance measures in terms of two indicators and some numerical examples are presented. In section 4, $M/M/c$ model with SWV model is analyzed. In section 5, we have given the comparison when the system follows MWV and SWV policies of the distribution of the number of customers.

2. Model Description

We consider an M/M/c queue with synchronous working vacation and impatient customers. The inter arrival times of the customers follows a Poisson process with arrival rate λ. The c servers serve the customers according to FCFB basis. An arriving customer has to wait in queue if he finds all the servers busy i.e. a queue begins to form when the number of servers is less than the number of customers in the system. The service times of each server follows an exponential distribution with rate μ during a non-vacation period, where we consider the stability that $\rho = \frac{\lambda}{c\mu} < 1$. If a server finds no customer in the system after completing serving a customer, all the servers immediately goes for a working vacation. The duration of working vacation for each server is exponentially distributed with parameter γ. If a customer arrives to a server during a working vacation period, it will serve the customer at an exponential rate η where $\eta < \mu$ i.e. the customer is served at a reduced service rate. When the servers return from their vacation and find the system non empty, they change their service rate from η to μ and a regular busy period starts. Otherwise, if the servers find no customer waiting in the queue after returning from their vacation, they immediately leave for another vacation.

If an arriving customer finds any of the c servers empty, it immediately gets service upon arrival. A customer has to wait in a queue, if all the servers are busy. A customer waiting in a queue becomes impatient if it finds all the servers in their working vacation period i.e. if it finds all the servers serving at rate η, the customer activates an impatient time T which is exponentially distributed with parameter γ and is independent of the customers

in the queue at that moment. The customer exits the queue and never returns if its service has not been completed before the time T expires. The inter arrival times, service times, vacation duration times and impatient time are all taken to be mutually independent.
To construct this system, we define a two dimensional continuous time discrete state Markov chain as $\{(L(t), J(t)), t \geq 0\}$ with state space

$$S = \{\{(0,0)\} \bigcup \{(n, j)\}, n \geq 1, j = 0,1\}$$

where L(t) denotes the total number of customers in the system at time t and J(t) denotes the State of the system at time t with

$$J(t) = \begin{cases} 1 & \text{when the servers are a non-vacation period at time t,} \\ 0 & \text{when the servers are in working vacation period at time t.} \end{cases}$$

3. The Stationary distribution

The steady state transition probabilities are defined by

$$P_{nj} = P\{L(t) = n, J(t) = j\},\ n \geq 0,\ j = 0,1$$

Then, we can have following set of balance equations as

$$\lambda P_{00} = (\eta + \xi)P_{1,0} + \mu P_{11}, \quad (1)$$

$$[\lambda + \gamma + n(\eta + \xi)]P_{n,0} = \lambda P_{n-1,0} + (n+1)(\eta + \xi)P_{n+1,0},\ if\ n \geq 1, \quad (2)$$

$$(\lambda + \mu)P_{11} = \gamma p_{1,0} + 2\mu P_{2,1}, \quad (3)$$

$$(\lambda + n\mu)P_{11} = \lambda P_{n-1,1} + (n+1)\mu P_{n+1,1} + \gamma P_{n,0}\ if\ 2 \leq n \leq c-1, \quad (4)$$

$$(\lambda + c\mu)P_{n,1} = \lambda P_{n-1,1} + c\mu P_{n+1,1} + \gamma P_{n,0}\ if\ n \geq c. \quad (5)$$

Define the (partial) probability generating functions

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$P_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1},$$

with $P_0(1) + P_1(1) = 1$ and $P_0'(z) = \sum_{n=1}^{\infty} nz^{n-1} P_{n,0}$.

Multiplying (2) with z^n and summing over n and rearrange terms, we get the differential equation

$$(\eta + \xi)(1-z)P_0'(z) = \left[\lambda(1-z) + \gamma\right]P_0(z) - (\gamma P_{0,0} + \mu P_{1,1}). \quad (6)$$

Similarly, multiplying (4) and (5) by z^n and summing over n,we get

$$(1-z)(\lambda z - c\mu)P_1(z) = \gamma z p_0(z) - (\gamma P_{0,0} + \mu P_{1,1})z + \mu(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \quad (7)$$

3.1. Solution of differential equation

Set

$$A=\gamma P_{0,0}+\mu P_{1,1}. \quad (8)$$

Then, for $z\neq 1$,

$$P_0'(z)-\left[\frac{\lambda}{\eta+\xi}+\frac{\gamma}{(\eta+\xi)(1-z)}\right]P_0(z)=-\frac{A}{(\eta+\xi)(1-z)}. \quad (9)$$

This is an ordinary linear differential equation with constant coefficients. To solve it, an integrating factor can be found as

$$\text{I.F}=e^{-\int\left[\frac{\lambda}{\eta+\xi}+\frac{\gamma}{(\eta+\xi)(1-z)}\right]dz}=e^{-\frac{\lambda z}{(\eta+\xi)}}(1-z)^{\frac{\gamma}{(\eta+\xi)}}$$

Hence the general solution to the differential equation (9) is given by

$$\frac{d}{dz}\left[e^{-\frac{\lambda z}{(\eta+\xi)}}(1-z)^{\frac{\gamma}{(\eta+\xi)}}P_0(z)\right]=\left[\frac{A}{(\eta+\xi)(1-z)}\right]e^{-\frac{\lambda z}{(\eta+\xi)}}(1-z)^{\frac{\gamma}{(\eta+\xi)}}. \quad (10)$$

Integrating from 0 to z, we get

$$P_0(z)=e^{\frac{\lambda z}{(\eta+\xi)}}(1-z)^{-\frac{\gamma}{(\eta+\xi)}}\left[P_0(0)-\frac{A}{(\eta+\xi)}\int_0^z e^{-\frac{\lambda z}{(\eta+\xi)}}(1-x)^{\frac{\gamma}{(\eta+\xi)}-1}dx\right]. \quad (11)$$

Then,

$$P_0(1)=e^{\frac{\lambda}{\eta+\xi}}\left[P_0(0)-\frac{A}{(\eta+\xi)}\int_0^1 e^{-\frac{\lambda z}{(\eta+\xi)}}(1-x)^{\frac{\gamma}{(\eta+\xi)}-1}dx\right]\lim_{z\to 1}(1-z)^{\frac{-\gamma}{\eta+\xi}}. \quad (12)$$

Since $0\leq P_0(1)=\sum_{n=0}^{\infty}P_{n,0}\leq 1$ and $\lim_{z\to 1}(1-z)^{-\frac{\gamma}{\mu_v+\xi}}=\infty$, so, we must have the term

$$P_{0,0}=P_0(0)=\frac{A}{(\eta+\xi)}K \quad (13)$$

where

$$K=\int_0^1 e^{-\frac{\lambda z}{(\eta+\xi)}}(1-x)^{\frac{\gamma}{(\eta+\xi)}-1}dx. \quad (14)$$

Define

$$Z(\lambda,\gamma)=-\lambda^{-\gamma}e^{-\lambda}(-\Gamma(\gamma,-\lambda)+\Gamma(\gamma)) \quad (15)$$

where $\Gamma(z)$ is the Γ function that has representation

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt \quad (16)$$

and

$$\Gamma(a,z)=\int_z^{\infty}e^{-t}t^{a-1}dt. \quad (17)$$

Some computations give

$$K=Z\left(\frac{\lambda}{\eta+\xi},\frac{\gamma}{\eta+\xi}\right). \quad (18)$$

From (8) and (13), we have

$$P_{0,0} = \left(\frac{\gamma P_{0,0} + \mu P_{1,1}}{\eta + \xi}\right)K = \frac{K\mu}{\eta + \xi - \gamma K} P_{1,1}. \qquad (19)$$

Substituting the value of A from (13) into (11), we obtain

$$P_0(z) = \frac{e^{\frac{\lambda z}{\eta+\xi}}}{(1-z)^{\frac{\gamma}{\eta+\xi}}}\left[1 - \frac{1}{K}\int_0^z e^{\frac{-\lambda x}{\eta+\xi}}(1-x)^{\frac{\gamma}{\eta+\xi}-1}dx\right]P_{0,0}. \quad (20)$$

Using L'Hospital's rule, we get

$$P_0(1) = \frac{(\eta+\xi)}{\gamma K} P_{0,0}. \qquad (21)$$

By substituting the value of $P_{0,0}$ from (19), we have the relation

$$\gamma P_0(1) = \gamma P_{0,0} + \mu P_{1,1}. \qquad (22)$$

Equation (7) can be written as follows:

$$P_1(z) = \frac{[\gamma P_0(z) - A]z}{(\lambda z - c\mu)(1-z)} - \frac{\mu}{\lambda z - c\mu} G(z), \qquad (23)$$

where

$$G(z) = \sum_{n=1}^{C} (c-n) P_{n,1} z^n. \, (24)$$

Equation (20) illustrates that $P_0(z)$ is a function of $P_{0,0}$, the proportion of time the servers are on working vacation and the system is empty. Also, $P_1(z)$ is a function of $P_0(z)$, A and G(z) as shown by (23). Thus, once $P_{0,0}$ and $P_{j,1}(j = 1, 2, ..., c)$ are obtained, $P_0(z)$ and $P_1(z)$ are completely determined.

3.2. Performance Measures

Applying L'Hospital's rule in (23), we have

$$P_1(1) = \frac{[\gamma P_0(1) - A] + \gamma P_0'(1)]}{c\mu - \lambda} + \frac{\mu}{c\mu - \lambda} G(1), \qquad (25)$$

where

$$G(1) = \sum_{n=1}^{c} (c-n) P_{n,1}. \qquad (26)$$

Applying (22) and (8) in (25), we have

$$P_1(1) = \frac{\gamma}{c\mu - \lambda} E(L_0) + \frac{\mu}{c\mu - \lambda} G(1). \qquad (27)$$

Applying L'Hopital's rule to (6), we have

$$E(L_0) = \lim_{z\to 1} P_0'(z) = \frac{-\lambda P_0(1) + \gamma P_0'(1)}{-(\eta+\xi)} = \frac{\lambda P_0(1) - \gamma E(L_0)}{(\eta+\xi)} \qquad (28)$$

implies that

$$P_0(1)=\frac{(\gamma+\eta+\xi)}{\lambda}E(L_0). \qquad (29)$$

Using (27) and (29) and noting that $P_0(1)+P_1(1)=1$, we obtain the expected number of customers during working vacation period as

$$E(L_0)=\frac{\lambda(1-\rho)}{\gamma+\eta(1-\rho)+\xi(1-\rho)}-\frac{\frac{\lambda}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1). \qquad (30)$$

Substituting (30) into (29), we get the probability that the system is in working vacation period as

$$P(J=0)=P_0(1)=\frac{(1-\rho)(\gamma+\eta+\xi)}{\gamma+\eta(1-\rho)+\xi(1-\rho)}-\frac{\frac{\gamma+\eta+\xi}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1) \quad (31)$$

and the probability that the system is in non-vacation period as

$$P(J=1)=P_1(1)=1-P_0(1)=\frac{\gamma\rho}{\gamma+\eta(1-\rho)+\xi(1-\rho)}+\frac{\frac{\gamma+\eta+\xi}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1). \qquad (32)$$

Now, we derive $E(L_1)$. Differentiating (23) and Using L'Hopital's rule, we get

$$E(L_1)=\lim_{z\to 1}P_1'(z)$$

$$=\lim_{z\to 1}\Big\{\frac{-z\lambda(-A+\gamma P_0(z))}{(1-z)(\lambda z-c\mu)^2}+\frac{-A+\gamma P_0(z))+z\gamma P_0'(z)}{(1-z)(\lambda z-c\mu)} \qquad (33)$$

$$+\frac{z(-A+\gamma P_0(z))}{(1-z)^2(\lambda z-c\mu)}+\mu\frac{[(c\mu-\lambda z)G'(z)+\lambda G(z)]}{(c\mu-\lambda z)^2}\Big\}$$

$$=\frac{\gamma(c\mu-\lambda)E(L_0(L_0-1))+2c\mu\gamma E(L_0)}{2(c\mu-\lambda z)^2}+\frac{G'(1)}{c(1-\rho)}+\frac{\rho G(1)}{c(1-\rho)^2}, \qquad (34)$$

where

$$G'(1)=\frac{dG(z)}{dz} \qquad at\ z=1$$

$$=\sum_{j=1}^{c} j(c-j)P_{j,1}. \qquad (35)$$

In order to get the value of $P_0''(1)$, we differentiate (6) twice on both sides such that

$$(\gamma+\xi)(1-z)P_0'''(z)+2\lambda P_0'(z)=[\lambda(1-z)+\gamma+2(\eta+\xi)]P_0''(z) \qquad (36)$$

where

$$P_0'''(z)=\frac{d^3P_0(z)}{dz^3}.$$

Letting $z=1$ in (36), we obtain

$$P_0''(1)=\frac{2\lambda}{\gamma+2(\eta+\xi)}P_0'(z). \qquad (37)$$

or, equivalently,

$$E[L_0(L_0-1)] = \frac{2\lambda}{\gamma+2(\eta+\xi)} E[L_0]. \qquad (38)$$

Substituting (38) into (34), we obtain the mean number of customers when the system is in regular busy period as

$$E[L_1] = \frac{\rho\gamma}{1-\rho}\left[\frac{1}{\gamma+2(\eta+\xi)} + \frac{1}{\lambda(1-\rho)}\right] E[L_0] + \frac{1}{c(1-\rho)} G'(1) + \frac{\rho}{c(1-\rho)^2} G(1). \qquad (39)$$

Hence, the mean number of customers in the system is

$$\begin{aligned} E[L] &= E[L_0] + E[L_1] \\ &= \left\{1 + \frac{\rho\gamma}{1-\rho}\left[\frac{1}{\gamma+2(\eta+\xi)} + \frac{1}{\lambda(1-\rho)}\right]\right\}\left[\frac{\lambda(1-\rho) - \frac{\lambda}{c}R(1)}{\gamma+\eta(1-\rho)+\xi(1-\rho)}\right] \\ &\quad + \frac{1}{c(1-\rho)} G'(1) + \frac{\rho}{c(1-\rho)^2} G(1). \qquad (40) \end{aligned}$$

Using (31) in (21), we finally get

$$\begin{aligned} P_{0,0} &= \frac{\gamma k}{\eta+\xi} P_0(1) \\ &= \frac{\gamma k}{\eta+\xi}\left[\frac{(1-\rho)(\gamma+\eta+\xi)}{\gamma+\eta(1-\rho)+\xi(1-\rho)} - \frac{\frac{\gamma+\eta+\xi}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)} G(1)\right]. \qquad (41) \end{aligned}$$

If the state of the system is $(n,1)$, then the service rates of the servers are $n\mu$ for $n \le c$ and $c\mu$ for $n > c$ respectively. Thus, the expected number of customers served per unit of time is given by

$$N_s = \sum_{n=1}^{c} n\mu P_{n,1} + \sum_{n=c+1}^{\infty} c\mu P_{n,1} = \mu[cP_1(1) - G(1)] \qquad (42)$$

which implies the proportion of customers served per unit of time is given by

$$P_s = \frac{N_s}{\lambda} = \frac{1}{c\rho}[cP_1(1) - G(1)] \qquad (43)$$

where $P_1(1)$ is given by (32).

When the state of the system is $(n,1)$, $n \ge 1$, the rate of customer abandonment of a customer due to impatience is $n\xi$. Hencethe mean rate of customer abandonment due to impatience is given by

$$R_a = \sum_{n=1}^{\infty} n\xi P_{n,0} = \xi E[L_0]. \qquad (44)$$

In this subsection we have expressed all the system performance measures in terms of $G(1)$ or/and $G'(1)$. In the next subsection, we compute these two indicators $G(1)$ and $G'(1)$.

3.3 Limiting Behavior

We consider the limiting behavior for some performance measures when $\rho \to 1$. Since $P_0(1) \geq 0$, therefore from (31), we get $0 \leq G(1) \leq c(1-\rho)$, which implies that

$$\lim_{\rho \to 1} G(1) = 0. \qquad (45)$$

Since $G(1) = \sum_{j=1}^{c} (c-j) P_{j,1}$, hence

$$\lim_{\rho \to 1} P_{j,1} = 0 \quad for \;\; j = 1, 2, ..., c-1. \qquad (46)$$

This gives

$$\lim_{\rho \to 1} G'(1) = \lim_{\rho \to 1} \sum_{j=1}^{c} j(c-j) P_{j,1} = 0. \qquad (47)$$

Fig.1 illustrates that $G(1)$ and $G'(1)$ approaches to zero when $\rho \to 1$, where the values of the parameters are ξ=0.3, γ=0.7, μ=3, η=1.5 and c=5. Equations (45) and (47) coincide with this observation.Using (45), we have from (31) and (32) that

$$\lim_{\rho \to 1} P_0(1) = 0, \qquad (48)$$

$$\lim_{\rho \to 1} P_1(1) = 1. \qquad (49)$$

Further, we get from (43) that

$$\lim_{\rho \to 1} P_s = 1. \qquad (50)$$

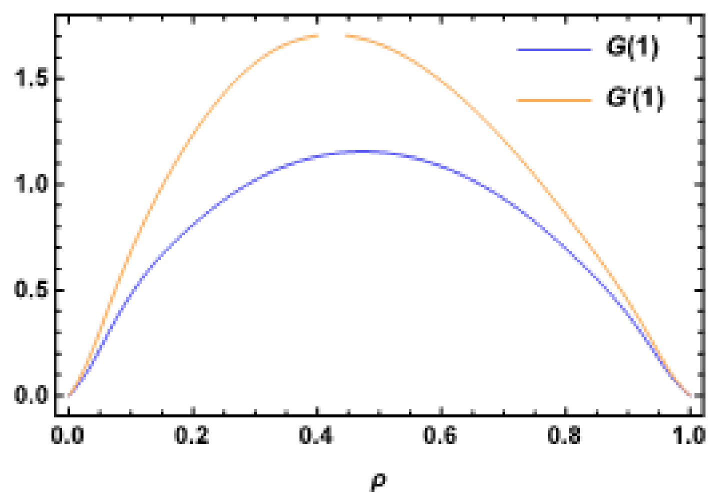


Figure 1: Effects of ρ on G (1) and $G'(1)$

3.4. Calculations of $\mathbf{G(1)}$ and $\mathbf{G'(1)}$

In order to compute $G(1)$ and $G'(1)$, we need to calculate the unknown probabilities $P_{j,1}$ for $j=1,2,...,c-1$. From (1), (2), (3) and (4), $P_{j,1}$ for $j=1,2,...,c-1$ and $P_{j,0}$ for $j=0,1,2,...,c-1$ satisfy the following \$2c-3\$ linear equations:

$$\lambda P_{00} = (\eta+\xi)P_{1,0} + \mu P_{11}, \tag{51}$$

$$[\lambda+\gamma+n(\eta+\xi)]P_{n,0} = \lambda P_{n-1,0} + (n+1)(\eta+\xi)P_{n+1,0},\ if\ 1 \le n \le c-2, \tag{52}$$

$$(\lambda+\mu)P_{11} = \gamma p_{1,0} + 2\mu P_{2,1}, \tag{53}$$

$$(\lambda+n\mu)P_{11} = \lambda P_{n-1,1} + (n+1)\mu P_{n+1,1} + \gamma P_{n,0},\ if\ 2 \le n \le c-2. \tag{54}$$

Therefore, we need another two independent equations to calculate all 2c-1 unknowns. From (19), we have

$$P_{0,0} = \frac{K\mu}{\eta+\xi-\gamma K}P_{1,1} \tag{55}$$

implies that

$$P_{1,1} = \delta P_{0,0}, \tag{56}$$

where

$$\delta = \frac{\eta+\xi-\gamma K}{K\mu_b}. \tag{57}$$

Substituting (22) into (31), we get

$$P_{0,0} + \frac{\mu}{\gamma}P_{1,1} = \frac{(c\mu-\lambda)(\gamma+\eta+\xi)}{c\mu\gamma+\eta(c\mu-\lambda)+\xi(c\mu-\lambda)} - \frac{\mu(\gamma+\eta+\xi)}{c\mu\gamma+\eta(c\mu-\lambda)+\xi(c\mu-\lambda)}G(1). \tag{58}$$

Hence, we get two more independent equations (56) and (58). Therefore, we obtain 2c-1 independent equations to solve for the $2c-1$ unknowns. We solve these equations analytically as follows.Substitute (56) into (51) and (53).

$$(\lambda-\mu\delta)P_{0,0} = (\eta+\xi)P_{1,0}, \tag{59}$$

$$(\lambda+\mu)\delta P_{0,0} = \gamma p_{1,0} + 2\mu P_{2,1}. \tag{60}$$

Thus, $P_{j,0}$, $j=1,2,...,c-1$, and $P_{j,1}$, $j=2,3,...,c-1$, satisfy(52), (54), (59), and (60). To write these equations in matrix form, we define two column vectors as follows:

$$T_0 = (P_{1,0}, P_{2,0}, ..., P_{(c-1),0})^T,$$

$$T_1 = (P_{2,1}, P_{3,1}, ..., P_{(c-1),1})^T. \tag{61}$$

Then, we have

$$PT_0 = SP_{0,0},$$

$$QT_0 + RT_1 = TP_{0,0}, \tag{62}$$

where P, Q and R are matrices given as follows:

$$P=\begin{pmatrix} (\eta+\xi) & 0 & 0 & \ldots & 0 & 0 & 0 \\ -a_1 & 2(\eta+\xi) & 0 & \ldots & 0 & 0 & 0 \\ \lambda & -a_2 & 3(\eta+\xi) & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & -a_{c-2} & (c-1)(\eta+\xi) \end{pmatrix},$$

$$Q=\begin{pmatrix} \gamma & 0 & \ldots & 0 & 0 \\ 0 & \gamma & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \gamma & 0 \end{pmatrix}, R=\begin{pmatrix} 2\mu & 0 & 0 & \ldots & 0 & 0 & 0 \\ -b_2 & 3\mu & 0\ldots & 0 & 0 & 0 & \\ \lambda & -b_3 & 4\mu & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & \lambda & -b_{c-2} & (c-1)\mu \end{pmatrix}, \quad (63)$$

where

$$a_n=\lambda+n\eta+\gamma,$$
$$b_n=\lambda+n\mu, \quad (64)$$

for $n=1,2,\ldots,c-2$ and S and T are two column vectors given as follows:

$$S=(\lambda-\mu\delta,-\lambda,0,\ldots,0)^T,$$
$$T=((\lambda+\mu)\delta,-\lambda\delta,0,\ldots,0)^T. \quad (65)$$

Clearly, matrices P and R are inverse matrices. Thus, from (62), we have

$$T_0=P^{-1}SP_{0,0},$$
$$T_1=R^{-1}(T-QP^{-1}S)P_{0,0}. \quad (66)$$

Let e_0 be a vector with $c-1$ elements all to be one and let e_1 be a column vector with $c-2$ elements all to be zero. Using (56) and (66), $G(1)$ can be written by

$$G(1)=(c-1)\delta P_{0,0}+FR^{-1}(T-QP^{-1}S)P_{0,0}, \quad (67)$$

where

$$F=(c-2,c-3,\ldots,1) \quad (68)$$

is a vector. Submitting (56) and (67) into (58), we can obtain $P_{0,0}$. The matrices P^{-1} and R^{-1} can be computed iteratively. Let $a_{i,j}$ denote the elements of matrix P^{-1} and let $b_{i,j}$ denote the elements of matrix R^{-1}. Then, we have

$$a_{i,j}=0,\ \ i<j,\ \ j=2,3,\ldots,c-1,$$

$$a_{j,j}=\frac{1}{j(\eta+\xi)},\ \ j=1,2,3,\ldots,c-1,$$

$$a_{i,j}=\frac{1}{i(\eta+\xi)}(a_{i-1}a_{(i-1),j}-\lambda a_{(i-2),j}),\ \ i>j,\ \ j=1,2,3,\ldots,c-1. \quad (69)$$

Since the matrix R has the same structure as the matrix P, using (69), it is easy to get the elements of R^{-1} as follows:

$$b_{i,j}=0,\ \ i<j,\ \ j=2,3,\ldots,c-1,$$

$$b_{j,j} = \frac{1}{(j+1)\mu}, \quad j = 1,2,3,...,c-1,$$

$$b_{i,j} = \frac{1}{(i+1)\mu}(b_i b_{(i-1),j} - \lambda b_{(i-2),j}), \;\; i > j, \quad j = 1,2,3,...,c-1. \tag{70}$$

Using (69) and (70), it is easy to get from (66) that

$$P_{1,0} = (\lambda - \mu\delta)a_{1,1}P_{0,0}, \tag{71}$$

$$P_{j,0} = [(\lambda - \mu\delta)a_{j,1} - \lambda a_{j,2}]P_{0,0}, \quad j = 2,3,...,c-1, \tag{72}$$

$$P_{j+1,1} = (b_1 b_{j,1} - \lambda b_{j,2})\delta P_{0,0} - \gamma\sum_{k=1}^{j} b_{j,k}P_{k,0}, \quad j = 1,2,...,c-2. \tag{73}$$

Define

$$\phi_0 = c - 1 + \sum_{j=1}^{c-2}(c-j-1)(b_1 b_{j,1} - \lambda b_{j,2}),$$

$$\phi_k = \sum_{j=k}^{c-2}(c-j-1)b_{j,k}, \; k = 1,2,...,c-2. \tag{74}$$

Then, using (73), we obtain

$$G(1) = \sum_{j=1}^{c-1}(c-j)P_{j,1} = V(\phi)P_{0,0}, \tag{75}$$

where

$$V(\phi) = \delta\phi_0 - \theta\sum_{k=1}^{c-2}\phi_k[(\lambda - \mu\delta)a_{k,1} - \lambda a_{k,2}]. \tag{76}$$

Define

$$\Psi_0 = c - 1 + \sum_{j=1}^{c-2}(j+1)(c-j-1)(b_1 b_{j,1} - \lambda b_{j,2}),$$

$$\Psi_k = \sum_{j=k}^{c-2}(j+1)(c-j-1)b_{j,k}, \; k = 1,2,...,c-2. \tag{77}$$

Using (73), we get

$$G'(1) = \sum_{j=1}^{c-1} j(c-j)P_{j,1} = V(\Psi)P_{0,0}, \tag{78}$$

where

$$V(\Psi) = \delta\Psi_0 - \gamma\sum_{k=1}^{c-2}\Psi_k[(\lambda - \mu\delta)a_{k,1} - \lambda a_{k,2}]. \tag{79}$$

Substituting (56) and (75) into (58), we obtain

$$P_{0,0} = \frac{k\gamma(\gamma+\eta+\xi)(c\mu_b - \lambda)}{(\eta+\xi)[c\mu\gamma + \eta(c\mu-\lambda) + \xi(c\mu-\lambda)] + k\gamma\mu(\gamma+\eta+\xi)V(\phi)}. \tag{80}$$

3.5. Numerical Results

In this section, we put forward some numerical illustrations for the results obtained in section 3.2. Figs 2 and 3 demonstrate the effects of vacation service rate η and impatient parameter ξ on $E(L)$ and $E(L_0)$ respectively. Evidently the server works faster with the increasing value of η, the expected system size $E(L)$ and the mean system size during the working vacation period $E(L_0)$ decreases when ξ is fixed. Also, we observe that $E(L)$ and $E(L_0)$ are bigger, when ξ is smaller. Fig. 4 demonstrates the state probability of the server and the server stays in non-vacation period i.e.P (J=1) obviously decreases with the increasing values of η. The probability that the server remains in vacation period P (J=0) increases, hence the utilization level of the system idle time also become larger. The state probability of the server also depends on the vacation rate γ. For example when γ =1.5, P (J=1) are evidently larger than those when γ = 0.5. It also shows that it is reasonable to establish the lower speed operation period or vacation period.

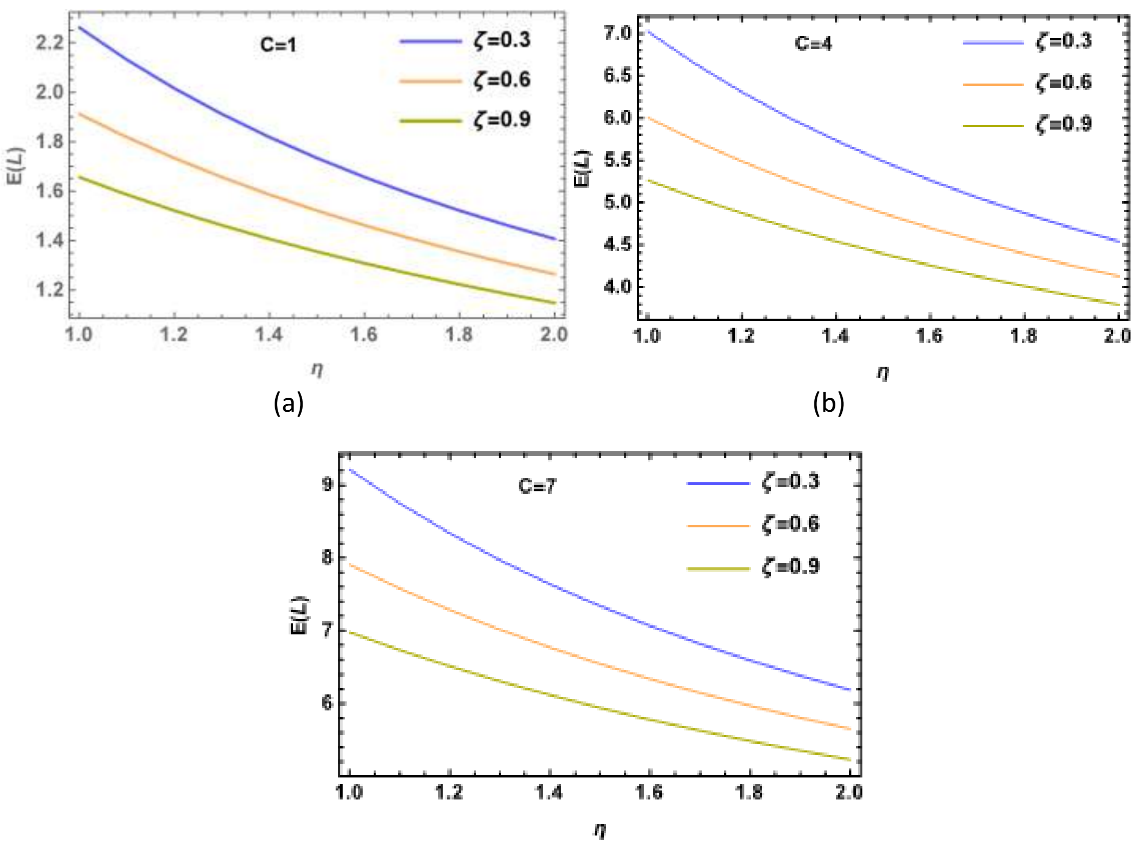
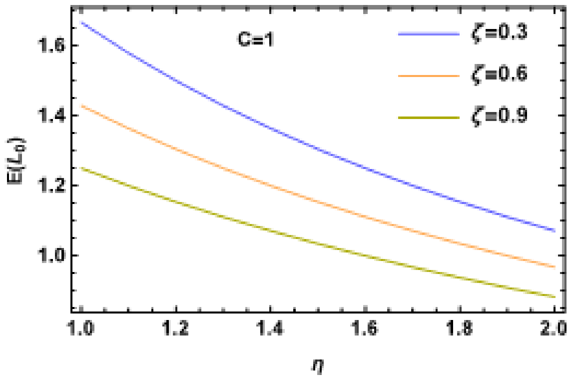
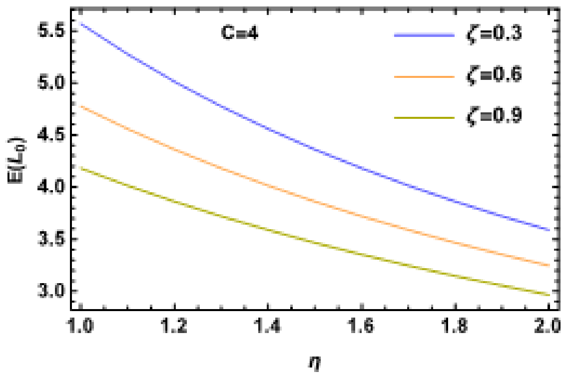


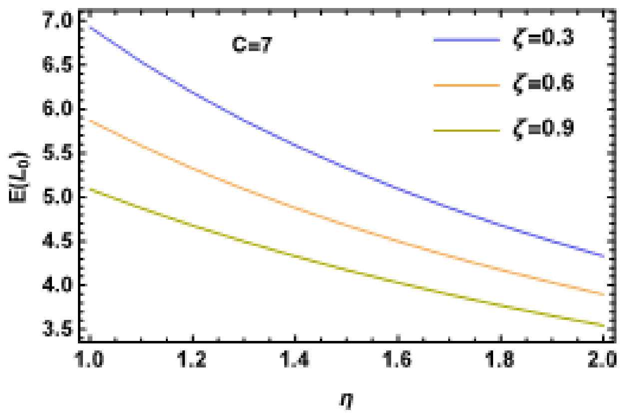
Figure 2: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6, \gamma = 0.2$ and $\mu = 5$.



(a)

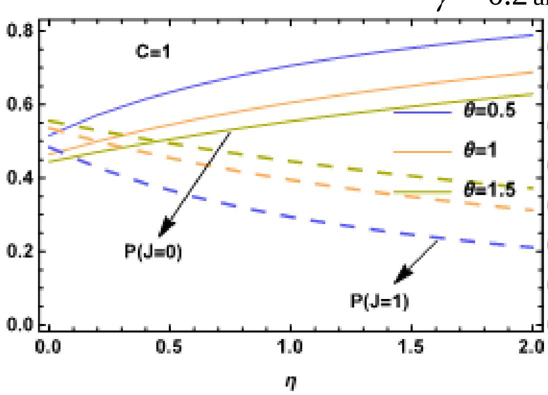


(b)

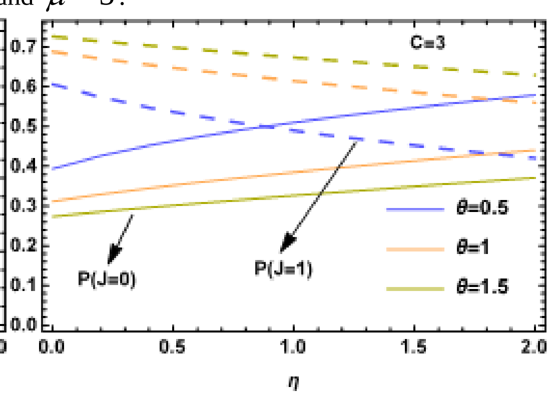


(c)

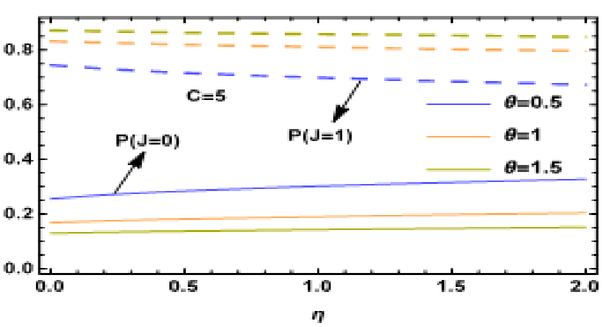
Figure 3: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.



(a)



(b)



(c)

Figure 4: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.

4. Single working vacation (SWV) model

The Single working vacation policy requires the server to take vacation at the instant when the there are no customers waiting in the queue. When the working vacation ends and the server finds the system non empty, it will change its service from η to μ and the regular busy starts. Otherwise, the server stays idle rather than taking another vacation and the new busy period starts when the first arrival occurs. Then the Markov chain $\{(L(t), J(t)), t \geq 0\}$ can be defined for SWV model as in MWV case, with state space

$S = \{\{(n, j)\}, n \geq 0, j = 0,1\}$,

$$J(t) = \begin{cases} 1 & \text{when the servers are in regular busy period or idle at time t ,} \\ 0 & \text{when the servers are in working vacation period at time t.} \end{cases}$$

We have the balance equations for the state transition probabilities given by

$$(\lambda+\gamma)P_{00} = (\eta+\xi)P_{1,0} + \mu P_{11}, \quad (81)$$

$$(\lambda+\gamma+n(\eta+\xi))P_{n,0} = \lambda P_{n-1,0} + (n+1)(\eta+\xi)P_{n+1,0}, \quad if\ n \geq 1, \quad (82)$$

$$\lambda P_{0,1} = \gamma p_{0,0}, \quad (83)$$

$$(\lambda+n\mu)P_{11} = \lambda P_{n-1,1} + (n+1)\mu P_{n+1,1} + \gamma P_{n,0}, \quad if\ 1 \leq n \leq c-1, \quad (84)$$

$$(\lambda+c\mu)P_{n,1} = \lambda P_{n-1,1} + c\mu P_{n+1,1} + \gamma P_{n,0}, \quad if\ n \geq c. \quad (85)$$

Define the (partial) probability generating functions

$$G_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$G_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1},$$

with $G_0(1) + G_1(1) = 1$ and $G_0'(z) = \sum_{n=1}^{\infty} nz^{n-1} P_{n,0}$.

Multiplying (82) with z^n and summing over n and rearrange terms, we get the differential equation

$$(\eta+\xi)(1-z)G_0'(z) = [\lambda(1-z)+\gamma]G_0(z) - \mu P_{1,1}. \quad (86)$$

Similarly, multiplying (84) and (85) by z^n and summing over n, we get

$$(1-z)(\lambda z - c\mu)G_1(z) = \gamma z G_0(z) - (\gamma P_{0,0} + \mu P_{1,1})z + z^2 \gamma P_{0,0} + \mu(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \quad (87)$$

Then,for $z \neq 1$,

$$G_0'(z) - \left[\frac{\lambda}{(\eta+\xi)} + \frac{\gamma}{(\eta+\xi)(1-z)}\right]G_0(z) = \frac{\mu P_{1,1}}{(\eta+\xi)(1-z)}. \quad (88)$$

Solving this differential equation, as in the MWV case, we get

$$G_0(z) = \frac{e^{\frac{\lambda z}{\eta+\xi}}}{(1-z)^{\frac{\gamma}{\eta+\xi}}}\left[1 - \frac{1}{K}\int_0^z e^{\frac{-\lambda x}{\eta+\xi}}(1-x)^{\frac{\gamma}{\eta+\xi}-1}dx\right]P_{0,0}. \quad (89)$$

Therefore, we get a similar expression for $G_0(z)$ as in MWV case. Here, we have

$$G_0(0) = P_{0,0} = \frac{\mu P_{1,1}}{\eta+\xi}K, \tag{90}$$

$$G_0(1) = \frac{(\eta+\xi)}{\gamma K}P_{0,0}. \tag{91}$$

From (90) and(91), we obtain

$$\gamma G_0(1) = \mu P_{1,1}. \tag{92}$$

Equation (87) can be written as follows:

$$G_1(z) = \frac{[\gamma G_0(z) - A]z + z^2\gamma P_{0,0}}{(\lambda z - c\mu)(1-z)} - \frac{\mu}{\lambda z - c\mu}G(z). \tag{93}$$

Equation (89) expresses $G_0(z)$ in terms of $P_{0,0}$, the proportion of time there are no costumers in the system and the servers are on working vacation. Also, equation (93) illustrates that $G_1(z)$ is a function of $G_0(z)$, A and G(z). Therefore, once $P_{0,0}$ and $P_{j,1}(j=1,2,...,c)$ are calculated, $G_0(z)$ and $G_1(z)$ are completely determined.

4.1. Performance Measures

Applying L'Hospital's rule to (93)and using (92), we have

$$G_1(1) = \frac{\gamma E(L_0) + \gamma(2-c)P_{0,0}}{c\mu - \lambda} + \frac{\mu}{c\mu-\lambda}G(1). \tag{94}$$

Applying L'Hospital's rule to (88), we have

$$E(L_0) = \lim_{z\to 1} G_0'(z) = \frac{\lambda G_0(1) - \gamma E(L_0)}{\eta+\xi}$$

implies that

$$G_0(1) = \frac{(\gamma+\eta+\xi)}{\lambda}E(L_0). \tag{95}$$

Therefore, we get a similar expression for $G_0(1)$ as in MWV case.

Using (94) and (95) and noting that $G_0(1)+G_1(1)=1$, we get the mean number of customers when the system is in working vacation as

$$E(L_0) = \frac{\lambda(1-\rho)}{\gamma+\eta(1-\rho)+\xi(1-\rho)} - \frac{\rho\gamma(2-c)P_{0,0}}{\gamma+\eta(1-\rho)+\xi(1-\rho)} - \frac{\frac{\lambda}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1). \tag{96}$$

Substituting (96) into (95), we have the probability that the server is in working vacation period as

$$P(J=0) = G_0(1) = \frac{(1-\rho)(\gamma+\eta+\xi)}{\gamma+\eta(1-\rho)+\xi(1-\rho)} - \frac{\rho\gamma(\gamma+\eta+\xi)(2-c)P_{0,0}}{\lambda[\gamma+\eta(1-\rho)+\xi(1-\rho)]}$$

$$-\frac{\frac{(\gamma+\eta+\xi)}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1) \quad (97)$$

and the probability that the server is in non-vacation period as

$$P(J=1)=G_1(1)=1-G_0(1)=\frac{\rho\gamma}{\gamma+\eta(1-\rho)+\xi(1-\rho)}+\frac{\rho\gamma(\gamma+\eta+\xi)(2-c)P_{0,0}}{\lambda[\gamma+\eta(1-\rho)+\xi(1-\rho)]}$$
$$+\frac{\frac{(\gamma+\eta+\xi)}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)}G(1). \quad (98)$$

Now, we derive $E(L_1)$. Differentiating (93) and using L'Hospital's rule, we get

$$E(L_1)=\lim_{z\to 1}G_1'(z)$$
$$=\lim_{z\to 1}\Big\{\frac{-\lambda[z(-A+\gamma G_0(z))+z^2\gamma P_{0,0}]}{(1-z)(\lambda z-c\mu)^2}+\frac{-A+\gamma G_0(z))+2z\gamma P_{0,0}+z\gamma G_0'(z)}{(1-z)(\lambda z-c\mu)}$$
$$+\frac{z(-A+\gamma G_0(z))+z^2\gamma P_{0,0}}{(1-z)^2(\lambda z-c\mu)}+\mu\frac{[(c\mu-\lambda z)G'(z)+\lambda G(z)]}{(c\mu-\lambda z)^2}\Big\}$$
$$=\frac{\gamma(c\mu-\lambda)E(L_0(L_0-1))+2c\mu\gamma E(L_0)+2\gamma[(c\mu-\lambda)-c\lambda]P_{0,0}}{2((c\mu-\lambda))^2}$$
$$+\frac{G'(1)}{c(1-\rho)}+\frac{\rho G(1)}{c(1-\rho)^2}. \quad (99)$$

In order to get the value of $G_0''(1)$, we differentiate (86)twice on both sides such that

$$E[L_0(L_0-1)]=\frac{2\lambda}{\gamma+2(\eta+\xi)}E[L_0]. \quad (100)$$

Substituting (100) into (99),we obtain

$$E[L_1]=\frac{\rho\gamma}{1-\rho}\Big\{\left[\frac{1}{\gamma+2(\eta+\xi)}+\frac{1}{\lambda(1-\rho)}\right]E[L_0]+\left[\frac{1}{\lambda}-\frac{1}{\mu(1-\rho)}\right]P_{0,0}\Big\}$$
$$+\frac{1}{c(1-\rho)}G'(1)+\frac{\rho}{c(1-\rho)^2}G(1). \quad (101)$$

Therefore, the mean number of customers in the system is

$$E[L]=E[L_0]+E[L_1]$$
$$=\left\{1+\frac{\rho\gamma}{1-\rho}\left[\frac{1}{\gamma+2(\eta+\xi)}+\frac{1}{\lambda(1-\rho)}\right]\right\}\left[\frac{\lambda(1-\rho)-\rho\gamma(2-c)P_{0,0}-\frac{\lambda}{c}G(1)}{\gamma+\eta(1-\rho)+\xi(1-\rho)}\right]$$
$$+\frac{\rho\gamma}{1-\rho}\left[\frac{1}{\lambda}-\frac{1}{\mu(1-\rho)}\right]P_{0,0}+\frac{1}{c(1-\rho)}G'(1)+\frac{\rho}{c(1-\rho)^2}G(1). \quad (102)$$

Using (97) in (91), we finally get

$$P_{0,0}=\frac{\gamma k}{\eta+\xi}G_0(1)$$

$$=\frac{\gamma k}{\eta+\xi}\left[\frac{\lambda(1-\rho)(\gamma+\eta+\xi)-\frac{\lambda(\gamma+\eta+\xi)}{c}}{\gamma+\eta(1-\rho)+\xi(1-\rho)+\frac{k\gamma^2\rho(2-c)(\gamma+\eta+\xi)}{\eta+\xi}}\right].$$

4.2. Calculations of $\mathbf{G(1)}$ and $\mathbf{G'(1)}$

In order to obtain $G(1)$ and $G'(1)$, we need to compute the unknown probabilities $P_{j,1}$ for $j=1,2,...,c-1$. Using (1),(2),(3) and(4), the probabilities $P_{j,1}$ for $j=1,2,...,c-1$ and $P_{j,0}$ for $j=0,1,2,...,c-1$ satisfy the following \$2c-3\$ linear equations:

$$(\lambda+\gamma)P_{00}=(\eta+\xi)P_{1,0}+\mu P_{11}, \tag{103}$$

$$[\lambda+\gamma+n(\eta+\xi)]P_{n,0}=\lambda P_{n-1,0}+(n+1)(\eta+\xi)P_{n+1,0},\ \ if\ 1\le n\le c-1, \tag{104}$$

$$\lambda P_{0,1}=\gamma p_{0,0}, \tag{105}$$

$$(\lambda+n\mu)P_{n,1}=\lambda P_{n-1,1}+(n+1)\mu P_{n+1,1}+\gamma P_{n,0},\ \ if\ 1\le n\le c-2. \tag{106}$$

Thus, we need two more independent equations to calculate all \$2c-1\$ unknowns.
From (90), we have

$$P_{0,0}=\frac{K\mu}{\eta+\xi}P_{1,1} \tag{107}$$

implies that

$$P_{1,1}=\delta P_{0,0} \tag{108}$$

where

$$\delta=\frac{\eta+\xi}{K\mu}. \tag{109}$$

Substituting the value of $G_0(1)$ from (92) into (97),we get

$$\frac{\mu}{\gamma}P_{1,1}=\frac{(c\mu-\lambda)(\gamma+\eta+\xi)}{c\mu\gamma+\eta(c\mu-\lambda)+\xi(c\mu-\lambda)}-\frac{\gamma(\gamma+\eta+\xi)(2-c)P_{0,0}}{c\mu\gamma+\eta(c\mu-\lambda)+\xi(c\mu-\lambda)}-\frac{\mu(\gamma+\eta+\xi)}{c\mu\gamma+\eta(c\mu-\lambda)+\xi(c\mu-\lambda)}G(1). \tag{110}$$

Thus we have two more independent equations (108) and (110).Hence, we have $2c-1$ independent equations to solve for the $2c-1$ unknowns. We solve these equations analytically as follows:

$$(\lambda+\gamma-\mu\delta)P_{0,0}=(\eta+\xi)P_{1,0}, \tag{111}$$

$$[-\gamma+(\lambda+\mu)\delta]P_{0,0}=\gamma p_{1,0}+2\mu P_{2,1}. \tag{112}$$

Thus, $P_{j,0}$, $j=1,2,...,\ c-1$, and $P_{j,1}$, $j=2,3,...,\ c-1$, satisfy(104), (106), (111), and (112). These equations can be written as equations in matrix form as follows
$PT_0=SP_{0,0}$,

$$QT_0+RT_1=TP_{0,0}. \tag{113}$$

Note that S and T are defined as follows:

$$S = (\lambda + \gamma - \mu\delta, -\lambda, 0, ..., 0)^T,$$
$$T = (-\gamma + (\lambda + \mu)\delta, -\lambda\delta, 0, ..., 0)^T. \quad (114)$$

Then as in MWV case, we get

$$P_{1,0} = (\lambda + \gamma - \mu\delta)a_{1,1}P_{0,0}, \quad (115)$$
$$P_{j,0} = [(\lambda + \gamma - \mu\delta)a_{j,1} - \lambda a_{j,2}]P_{0,0}, \quad j = 2, 3, ..., c-1, \quad (116)$$
$$P_{j+1,1} = (b_1 b_{j,1} - \lambda b_{j,2})\delta P_{0,0} - \gamma b_{j,1}P_{0,0} - \gamma\sum_{k=1}^{j} b_{j,k}P_{k,0} \quad , j = 1, 2, ..., c-2. \quad (117)$$

Using (117), we obtain

$$G(1) = \sum_{j=1}^{c-1}(c - j)P_{j,1} = V'(\phi)P_{0,0}, \quad (118)$$
$$G'(1) = \sum_{j=1}^{c-1} j(c - j)P_{j,1} = V'(\Psi)P_{0,0}, \quad (119)$$

where

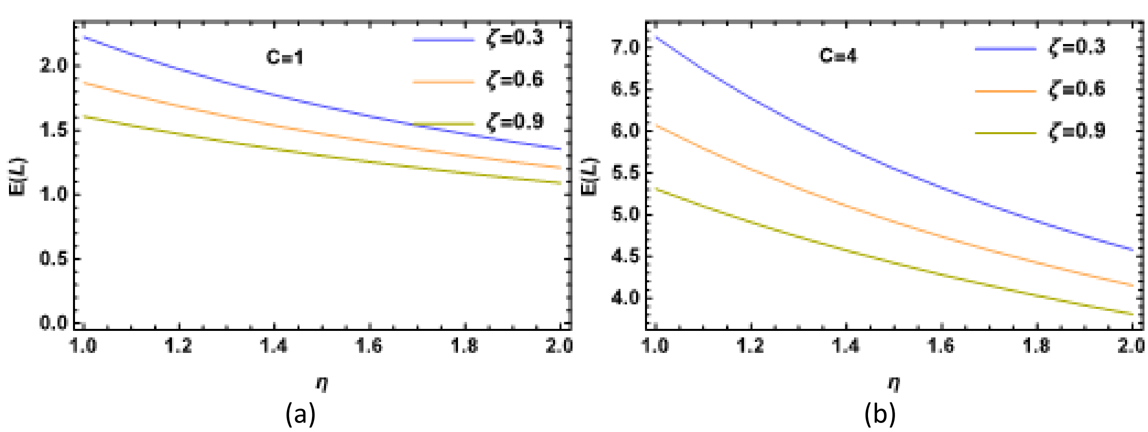
$$V'(\phi) = \delta\phi_0 - \gamma\sum_{j=1}^{c-2}(c - j - 1)b_{j,1} - \gamma\sum_{k=1}^{c-2}\phi_k[(\lambda + \gamma - \mu\delta)a_{k,1} - \lambda a_{k,2}], \quad (120)$$
$$V'(\Psi) = \delta\Psi_0 - \gamma\sum_{j=1}^{c-2}(j+1)(c - j - 1)b_{j,1} - \gamma\sum_{k=1}^{c-2}\Psi_k[(\lambda + \gamma - \mu\delta)a_{k,1} - \lambda a_{k,2}]. \quad (121)$$

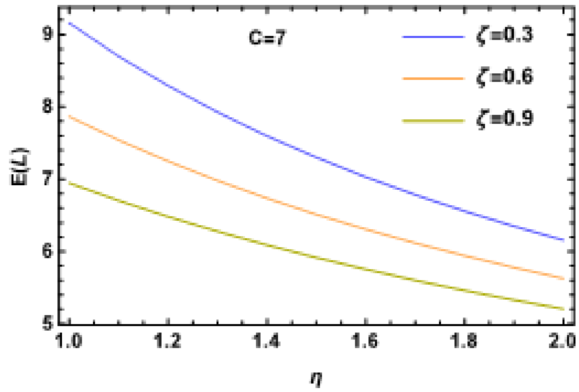
Substituting (108) and (118) into (110), we obtain

$$P_{0,0} = \frac{k\gamma(\gamma + \eta + \xi)(c\mu - \lambda)}{(\eta + \xi)[c\mu\gamma + \eta(c\mu - \lambda) + \xi(c\mu - \lambda)] + k\gamma^2(\gamma + \eta + \xi)(2 - c) + k\gamma\mu(\gamma + \eta + \xi)V'(\phi)}. \quad (122)$$

4.3. Numerical Results

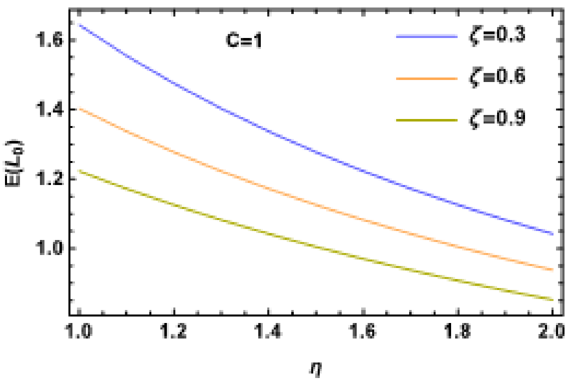
This section presents some numerical examples to show the effect of system performance measures obtained in section 4.1. Figs 5 and 6 shows that $E(L)$ and $E(L_0)$ decreases with increasing value of η when ξ is fixed and their values are bigger when ξ is small. Fig.7 shows that P (J=1) and P (J=0) decreases and increases respectively with the increasing values of η.



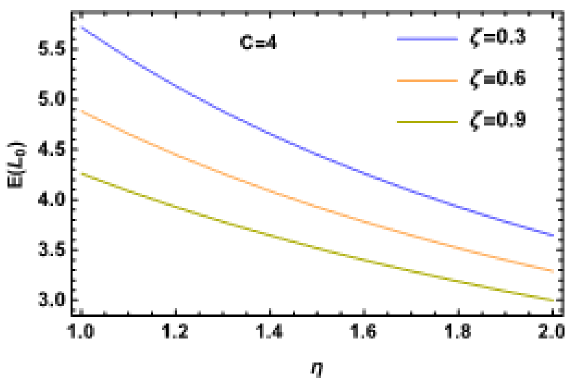


(c)

Figure 5: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.



(a)



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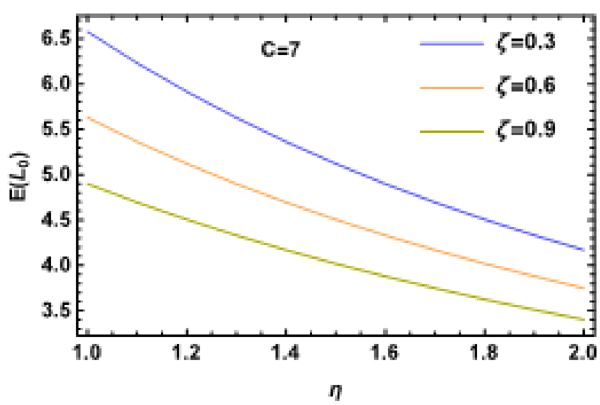


(c)

Figure 6: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.

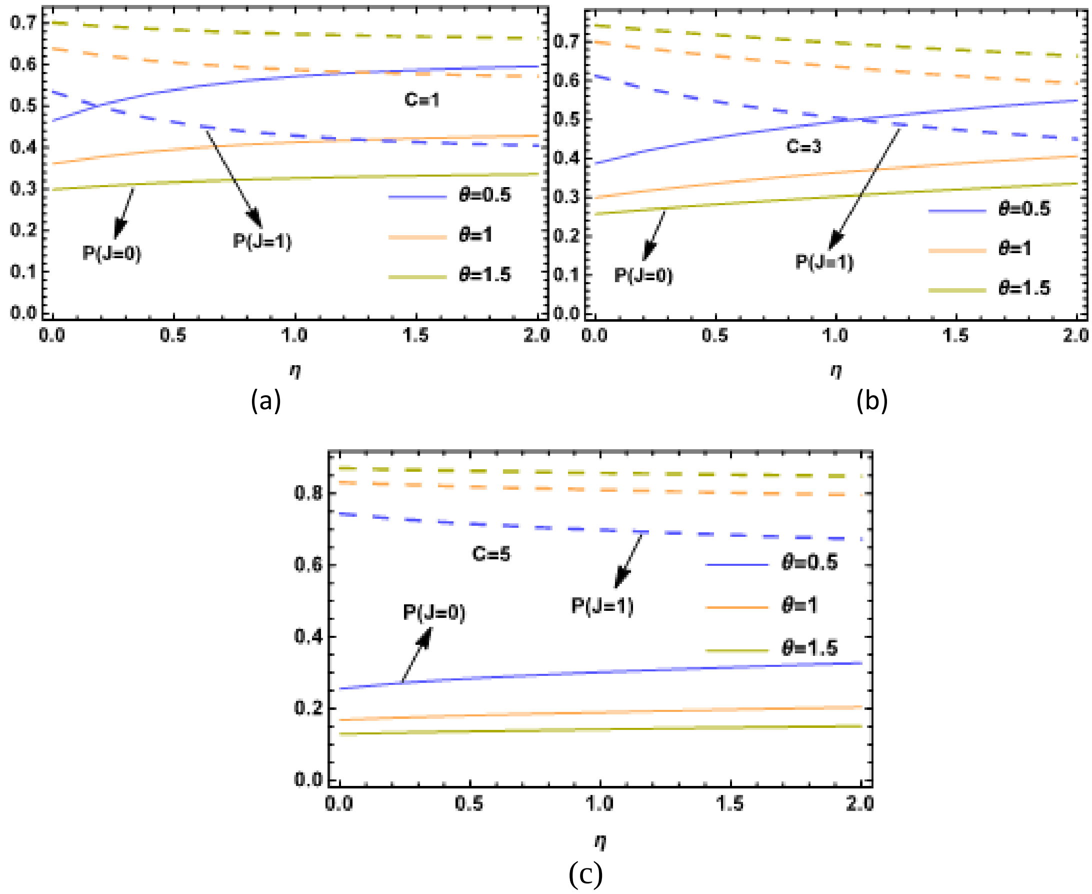
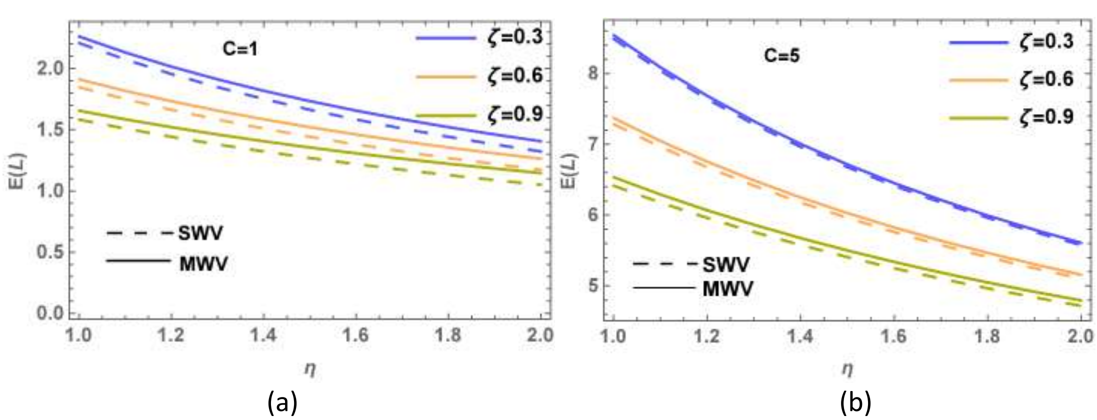
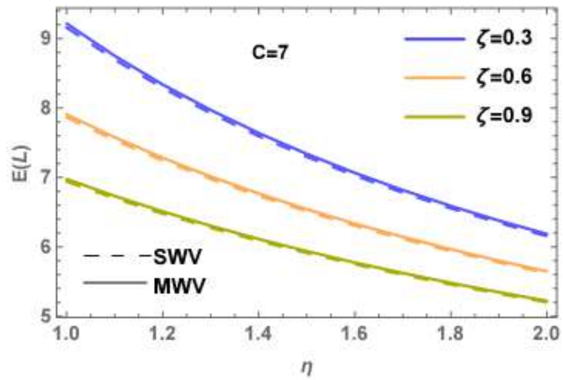


Figure 7: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.

5. Comparison

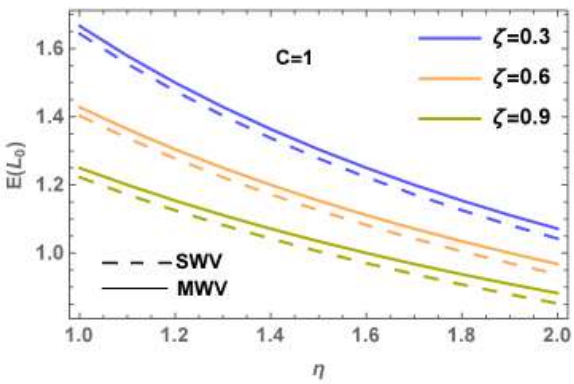
This section gives the comparison between the MWV model and SWV model when ξ is fixed in terms of their mean system size E[L] and mean system size during working vacation $E[L_0]$. From figs.8 and 9, it is clear that SWV model is more efficient compared to the MWV model in the sense that mean system size E[L] and mean system size during working vacation period $E[L_0]$ in SWV is always less than that in the MWV model.



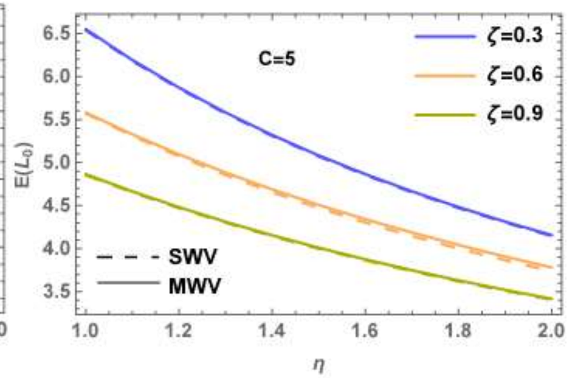


(c)

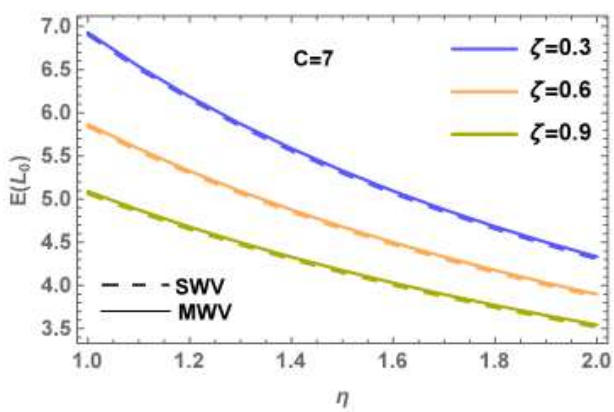
Figure 8: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.



(a)



(b)



(c)

Figure 9: Mean queue length E[L] versus service rate η in working vacation period when $\rho = 0.6$, $\gamma = 0.2$ and $\mu = 5$.

6. Conclusion

We have analyzed the synchronous working vacation policy in an M/M/c queuing model with impatient customers. We have discussed two types of WV policies i.e. the multiple working vacations (MWV) policy and single working vacation (SWV) policy. Explicit expressions for some system performance measures have been derived in terms of two indicators $G(1)$ and $G'(1)$. We have given some numerical illustrations which demonstrate that the above derived theoretical results are reasonable and can be directly used to solve the practical problems. This work emphasizes the fact that when ξ is fixed, mean queue length E (L) and mean queue length during working vacation period $E(L_0)$ in SWV model is less than that of MWV model. Hence, the efficiency of SWV model is more as compared to the MWV model.

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