A New Five-Parameter Fréchet Model for Extreme Values

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Abstract

A new five parameter Fréchet model for Extreme Values was proposed and studied. Various mathematical properties including moments, quantiles, and moment generating function were derived. Incomplete moments and probability weighted moments were also obtained. The maximum likelihood method was used to estimate the model parameters. The flexibility of the derived model was accessed using two real data set applications.

Keywords: Fréchet, Weibull-G, Moments, Hazard rate, Hazard rate, Maximum Likelihood.

1. Introduction

The extreme value theory is a very important theory in statistics it was devoted to stochastically series of independent and identical distributed variables. In other words, one can say it was devoted to the study of the behavior of extreme values, even though these values have a very low chance to appear, they can turn out to have a very high impact to the observed system. Finance and insurance are the best fields of research to observe the importance of extreme events. The extreme value theory can be considered as a developing area of research. It has been started in the last century as an equivalent theory to the central limit theory, which is dedicated to studying the asymptotic distribution of the average of a sequence. The central limit theorem states that the sum and the mean of an arbitrary finite distribution are normally distributed under the condition that the sample size is sufficiently large. However, in some practical studies, we are looking for the limiting distribution of maximum or minimum values rather than the average of the data. Assume that $X_1, X_2, \ldots, X_n$ is a sequence of *iid* random variables distributed with cumulative distribution function (cdf) denote F(x). One of the most interesting statistics in research is the sample maximum.

$$M_n = \max\{X_1, X_2, \ldots, X_n\}. \quad (1)$$

This theory studied the behavior of (1) as the sample size n increases to infinity.

$$p_r\{M_n \leq x\} = p_r\{M_1 \leq x\}\, p_r\{M_2 \leq x\} \ldots .\, p_r\{M_n \leq x\} = F(x)^n.$$

Suppose there are sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$p_r\left\{\frac{(M_n - b_n)}{a_n} \leq x\right\} \to G(x) \quad as\ n \to \infty. \tag{2}$$

Then if G(x) is a non-degenerate distribution function then it will belong to one of the three following fundamental types of classic extreme value family

1. Type-I (Gumbel distribution).
2. Type-II (Fréchet distribution).
3. Type-III (Weibull distribution).

The extreme value theory focuses on the behavior of block maxima or minima. The extreme value theory was introduced first by Fréchet (1927) and Fisher and Tippett (1928) then followed by Von Mises (1936) and completed by Gnedenko (1943), Von Mises (1964), Kotz and Johnson (1992), among others. The Fréchet (`Fr' for short) distribution is one of the important distributions in extreme value theory, and it has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves. For more details about the Fr distribution and its applications, see Kotz and Nadarajah (2000). Moreover, applications of this distribution in various fields are given in Harlow (2002). Recently, some extensions of the Fréchet distribution are considered. The exponentiated Fréchet by Nadarajah and Kotz (2003), beta Fréchet by Nadarajah and Gupta (2004), Nadarajah and Kotz (2008) and Zaharim et al. (2009), beta Fréchet by Barreto-Souza et al. (2011) and Mubarak (2013), transmuted Fréchet by Mahmoud and Mandouh (2013), Marshall-Olkin Fréchet by Krishna et.al. (2013), gamma extended Fréchet by da Silva et al. (2013), transmuted exponentiated Fréchet by Elbatal et al. (2014), transmuted Marshall-Olkin Fréchet by Afify et al. (2015), transmuted exponentiated generalized Fréchet by Yousof et al. (2015), beta exponential Fréchet by Mead et al. (2016), Kumaraswamy Marshall-Olkin Frèchet by Afify et al. (2016b), Weibull Fréchet by Afify et al. (2016b), Kumaraswamy transmuted Marshall-Olkin Fréchet by Yousof et al. (2016) and beta transmuted Fréchet by Afify et al. (2016c). The probability density function (pdf) and cdf of the Fréchet (Fr) distribution are given by (for X>0)

$$g(x;\,\gamma,\beta) = \beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right], \tag{3}$$

and

$$G(x;\,\gamma,\beta) = \exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right], \tag{4}$$

respectively, where $\gamma > 0$ is a scale parameter and $\beta > 0$ is a shape parameter, the pdf of the WFr distribution is given (for x>0) by

$$g(x) = ab\beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-b\left(\frac{\gamma}{x}\right)^{\beta}\right]\left(1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right)^{-(b+1)}\exp\left[-a\left(\exp\left(\frac{\gamma}{x}\right)^{\beta}-1\right)^{-b}\right], \tag{5}$$

where γ and a are scale parameters, b, and β are the shape parameters representing various shapes of WFr distribution. Its cdf under the condition of non-negativity of the parameters can be expressed as

$$G_{WF}(x; a, b\,\gamma, \beta) = 1 - \exp\left[-a\left(\exp\left(\frac{\gamma}{x}\right)^{\beta} - 1\right)^{-b}\right]. \tag{6}$$

In this article, we introduce an extension of Fr model using the WFr model and the transmuted-G (TG) family of distributions proposed by Shaw and Buckley (2007).

2. The TWFr Distribution

The cdf of the transmuted Weibull Fréchet (TWFr) distribution can be expressed as

$$F(x)=\left(1-\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right)\left[1+\lambda\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right], \tag{7}$$

and $|\lambda|\leq 1$ is additional shape parameter and a is scale parameter. Henceforward, we will consider a=1 except the application part, the corresponding pdf of (7)

$$f(x)=b\beta\gamma^{\beta}x^{-(\beta+1)}\frac{\exp\left[-b\left(\frac{\gamma}{x}\right)^{\beta}\right]\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right\}}{\left\{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{b+1}\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}}, x>0, \tag{8}$$

the reliability function for the TWFr distribution can be expressed as

$$R(x)=1-\left(1-\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right)\left[1+\lambda\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right], \tag{9}$$

the hazard rate function for the TWFr distribution can be expressed as

$$\tau(x)=\frac{b\beta\gamma^{\beta}x^{-(\beta+1)}\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right\}}{\left\{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{b+1}\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}}}{1-\left(1-\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right)\left[1+\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right]}, x>0. \tag{10}$$

and cumulative hazard rate function

$$H(x)=-\ln\left\{1-\left(1-\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right)\left[1+\lambda\exp\left\{-a\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}\right]\right\}, \tag{11}$$

Below is a simple motivation for the development of TWFr distribution. Suppose "T_1 and T_2" be two independent random variables from cdf in (7). Define

$$X=\begin{cases}T_{1:2} & \text{with probability } \frac{1}{2}(1+\lambda);\\ T_{2:2} & \text{with probability } \frac{1}{2}(1-\lambda);\end{cases}$$

Where $T_{1:2}=\min\{T_1,T_2\}$ and $T_{2:2}=\max\{T_1,T_2\}$, then the cdf of X is given by (7).

Figure 1 and 2 give some plots of p.d.f. and h.r.f. of TWFr distribution for some parameter values.

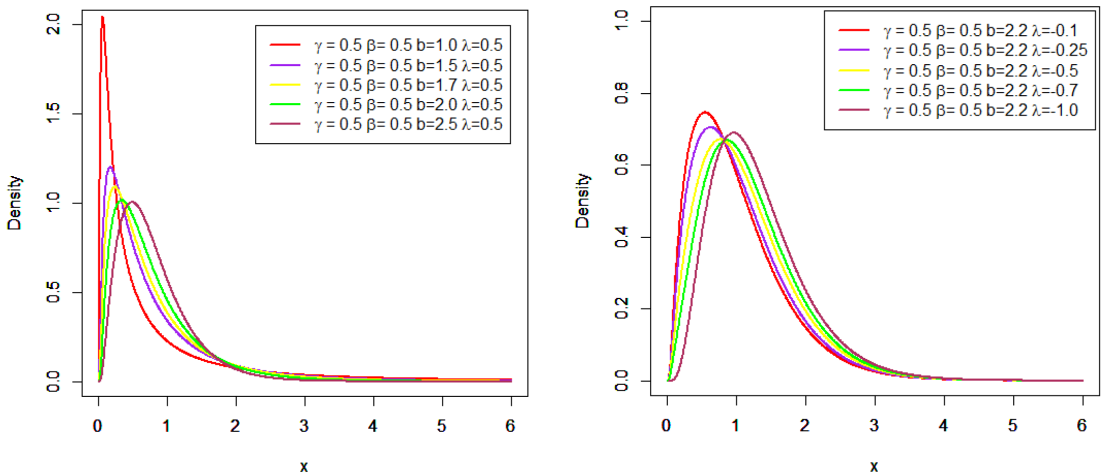


Figure 1: Plots of the TWFr pdf for some parameter values

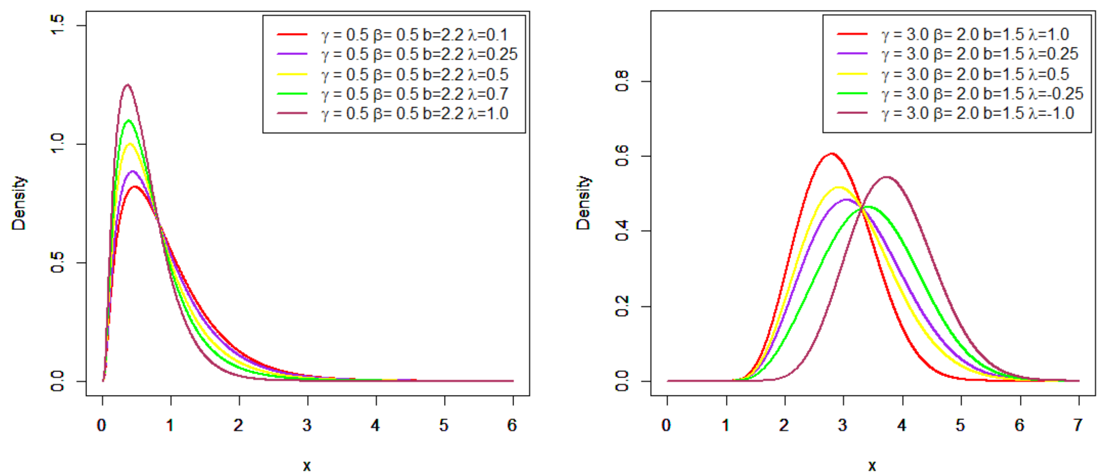


Figure 2: Plots of the TWFr pdf for some parameter values

3. Mixture Representation

The TWFr density function given in Eq. (8) can be expressed as

$$F(x;\ \lambda, b, \gamma, \beta) = 1 + (\lambda - 1)\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\} - \lambda\exp\left\{-2\left[\frac{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}\right]^{b}\right\}, \quad (12)$$

and after some algebra, we have

$$F(x) = 1 + \sum_{i,j=0}^{\infty} v_{i,j}\, H_{bi+j}(x) \quad (13)$$

where $H_{bi+j}(x)$ is the Fr cdf with scale parameter $\gamma[bi+j]^{1/\beta}$ and shape parameter β.

$$v_{i,j} = \frac{(-1)^{i+j}}{i!}\binom{-\alpha i}{j}\left(\lambda - 1 - \lambda\, 2^{i}\right),$$

the corresponding TWFr density function is obtained by differentiating (15)

$$f(x) = \sum_{i,j=0}^{\infty} v_{i,j}\, b_{bi+j}(x), \qquad (14)$$

where $h_{bi+j}(x)$ is the Fr cdf with scale parameter $\gamma[bi+j]^{1/\beta}$ and shape parameter β.

Let $c = \inf\left\{x \middle| \left\{\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\} > 0\right\}$. Then the asymptotics of cdf, pdf and hrf as $x \to c$ are given by

$$F(x) \sim (1+\lambda)\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right] \qquad \text{as} \qquad x \to c,$$

$$f(x) \sim b(1+\lambda)\beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right] \qquad \text{as} \qquad x \to c,$$

and

$$h(x) \sim b(1+\lambda)\beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right] \qquad \text{as} \qquad x \to c.$$

The asymptotic of cdf, pdf and hrf when $x \to \infty$ are given by

$$1 - F(x) \sim \exp\left(-\left\{1 - \exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{-b}\right) \qquad \text{as} \qquad x \to \infty,$$

$$f(x) \sim \frac{b\beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]}{\left\{1 - \exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{b+1}}\exp\left(-\left\{1 - \exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{-b}\right) \qquad \text{as} \qquad x \to \infty,$$

and

$$h(x) \sim b\beta\gamma^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{\gamma}{x}\right)^{\beta}\right]\right\}^{-b-1} \qquad \text{as} \qquad x \to \infty.$$

4. Mathematical properties

4.1 Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The $(s,r)th$ PWMs of X following the TWFr model, say $\rho_{s,r}$ is formally defined by

$$\rho_{s,r} = E\{X^{s}F(X)^{r}\} = \int_{-\infty}^{\infty} x^{s}F(x)^{r}f(x)dx.$$

Using equations (pdf) and (cdf), we can write

$f(x)F(x)^{r} = \sum_{i,j=0}^{\infty} p_{i,j}\, h_{b(i+1)+j}(x),$

where $h_{b(i+1)+j}(x)$ is the Fr density with scale parameter $\gamma[b(i+1)+j]^{1/\beta}$ and shape parameter β. and

$$p_{i,j} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+i+j}(h+1)^i \left[(1+\lambda)\binom{r+k}{h} - 2\lambda\binom{r+k+1}{h}\right]}{i!\, b^{-1}\lambda^{-k}(1+\lambda)^{k-r}[b(i+1)+j]} \binom{r}{k}\binom{-[b(i+1)+1]}{j}.$$

Then, the $(s,r)th$ PWMs of X can be expressed as

$$\rho_{s,r} = \sum_{i,j=0}^{\infty} \frac{p_{i,j}}{\gamma^{-r}}[b(i+1)+j]^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right).$$

4.2 Moments, incomplete moments and generating function

The r^{th} ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$. Then we obtain

$$\mu'_r = \sum_{i,j=0}^{\infty} \frac{v_{i,j}}{\gamma^{-r}}[ai+j]^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right). \qquad (15)$$

Setting $r=1$, we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Then n^{th} central moments of X, say M_n, follows as $M_n = E(X-\mu)^n = \sum_{h=0}^{n}(-1)^h\binom{n}{h}(\mu'_1)^n\mu'_{n-h}$. The cumulants (k_n) of X follow recursively from $k_n = \mu'_n - \sum_{r=0}^{n-1}(-1)^h\binom{n-1}{r-1}k_r\mu'_{n-h}$, where $k_1 = \mu'_1, k_2 = \mu'_2 - {\mu'_1}^2, k_3 = \mu'_3 - 3\mu'_2\mu'_1 + {\mu'_1}^3$, etc. The skewness and kurtosis measures also can be calculated from the ordinary moments using well-known relationships. The main applications of the first incomplete moments refer to the mean deviations and Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The rth incomplete moment, say $\varphi_r(t)$, of X can be expressed from (10) as

$$\varphi_r(t) = \int_{-\infty}^{t} x^r f(x)dx = \sum_{i,j=0}^{\infty} \frac{v_{i,j}}{\gamma^{-r}}[ai+j]^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}, [ai+j]-\left(\frac{\gamma}{t}\right)^{\beta}\right). \qquad (16)$$

The mean deviations about the mean $[\delta_1 = E(|X-\mu'_1|)]$ and about the median $[\delta_2 = E(|X-M|)]$ of X are given by $\delta_1 = 2\mu'_1 F(\mu'_1) - 2\varphi_1(\mu'_1)$ and $\delta_2 = \mu'_1 - 2\varphi_1(M)$,respectively, where $\mu'_1 = E(X)$, $M = median(X) = Q(0.5)$ is the median, $F(\mu'_1)$ is easily calculated from (5) and $\varphi_1(t)$ is the first incomplete moment given by (12) with r=1. Here, we provide two formulae for the moment generating function (mgf) $M_x(t) = E(e^{tX})$ of X. Clearly, the first one can be derived from equation (9), for $r < b$, as

$$M_x(t) = \sum_{k=0}^{\infty} \Upsilon_k\, M_{[(\alpha+k)\theta]}(t) = \sum_{k=0}^{\infty} \Upsilon_k \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r = \sum_{k,r=0}^{\infty} \frac{\Upsilon_k\, t^r}{a^{-r} r!}[(\alpha+k)\theta]^{\frac{r}{b}}\,\Gamma\left(1-\frac{r}{\beta}\right).$$

A second formula for $M_x(t)$. Setting $y = x^{-1}$ in (3), we can write this mgf $M(t; a, b) = ba^b \int_0^\infty \exp\left(\frac{t}{y}\right) y^{(b-1)} \exp\{-(ay)^b\}$. By expanding the first exponential and calculating the integral, we have

$$M(t;\, \gamma, \beta) = b\gamma^b \int_0^\infty \sum_{m=0}^{\infty} \frac{t^m}{m!} \exp\left(\frac{t}{y}\right) y^{\beta-m-1} \exp\{-(\gamma y)^\beta\} = \sum_{m=0}^{\infty} \frac{\gamma^m\, t^m}{m!}\, \Gamma\left(\frac{\beta - r}{\beta}\right),$$

where the gamma function is well-defined for any non-integer b. Consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q \left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_1, A_p) \\ (\beta_1, B_1), \dots, (\beta_1, B_p)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{x^n}{n!}$$

Then, we can write $M(t;\, \gamma, \beta)$ as

$$M(t;\, \gamma, \beta) = \; {}_1\Psi_0 \left[\begin{matrix}(1 - \beta^{-1}) \\ -\end{matrix}; \gamma t\right]. \tag{17}$$

Combining expressions (17) and the above equation, we obtain the mgf of X, say $M(t)$, as

$$M(t;\, \gamma, \beta) = \sum_{i,j=0}^{\infty} v_{i,j} \quad {}_1\Psi_0 \left[\begin{matrix}((1 - \beta^{-1}) \\ -\end{matrix}; \gamma [bi + j]^{\frac{1}{\beta}} t\right].$$

4.3 Order statistics

Let $X_1, X_2, \dots, X_n$ be random sample from the TWFr model of distributions and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$be the corresponding order statistics. The pdf of ith order statistics, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x),$$

where $B(.,.)$ is the beta function. Substituting (5) and (6) in equation (13) and using a power series expansion, we get

$$t_{m,w} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+m+w} \left[(1+\lambda)\binom{j+i+k-1}{h} - 2\lambda \binom{j+i+k}{h}\right]}{i!\, b^{-1}\, \lambda^{-k} (h+1)^{-m} (1+\lambda)^{k-(j+i-1)} [b(m+1)+w]} \binom{j+i-1}{k} \binom{-[b(m+1)+1]}{w}.$$

The pdf of $Xi{:}n$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \sum_{j=0}^{n-i} t_{m,w} h_{b(m+1)+w}\,(x).$$

Where $h_{b(m+1)+w}(x)$ is the Fr density with scale parameter $\gamma[b(m+1)+w]^{\frac{1}{\beta}}$ and shape parameter β. Based on the last equation, we note that the properties of $Xi{:}n$ follow from those properties of Y_{k+1}. For example, the moments of $Xi{:}n$ can be expressed as

$$E(X_{i:n}^q) = \sum_{m,w=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j} t_{m,w}}{B(i, n-i+1)\gamma^{-q}} [b(m+1)+w]^{\frac{q}{\beta}} \Gamma\left(1 - \frac{q}{\beta}\right). \tag{18}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (18), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TWFr order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E(X_{r-d:r}),\quad r \geq 1.$$

4.4 Moments of the residual and reversed residual life

Let X be a random variable usually representing the life length for a certain unit at age t (where this unit can have multiple interpretations), then the random variable $X_t = X - t|X > t$ represents the remaining lifetime beyond that age. Moreover, the nth moment of the residual life, denoted by

$m_n(x) = E\{(X-x)^n|X > x\},\ n = 1,2,\ldots$ uniquely determines F(x), The nth moment of the residual life of X is given by $M_n(t) = \frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n dF(x)$. Then, we ca write (for $r < \beta$□)

$$m_n(t) = \frac{1}{1-F(t)}\sum_{r=0}^{n}\frac{(-1)^{n-r}t^{n-r}n!}{r!\Gamma(n-r+1)}\sum_{i,j=0}^{\infty}\frac{v_{i,j}}{\gamma^{-r}}(bi+j)^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta},(bi+j)\left(\frac{\gamma}{t}\right)^{\beta}\right).$$

The nth moment of the reversed residual life, say $M_n(t) = E\{(t-X)^n|X \leq t\}$ for $t > 0$ and $n = 1,2,\ldots$ uniquely determines F(x) (Navarro et al. 1998). We obtain $M_n(t) = \frac{1}{F(t)}\int_0^t(t-x)^n dF(x)$. Therefore, the nth moment of the reversed residual life of X given that $r < \beta$ becomes

$$M_n(t) = \frac{1}{F(t)}\sum_{r=0}^{n}\frac{(-1)^j n!}{r!\,(n-r)!}\sum_{i,j=0}^{\infty}\frac{v_{i,j}}{\gamma^{-r}}(bi+j)^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta},(bi+j)\left(\frac{\gamma}{t}\right)^{\beta}\right).$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is defined by $M_1(t) = E\{(t-X)|X \leq t\}$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0; x). The MRRL of X can be obtained by setting n = 1 in the above equation.

4.5 Stress-strength model

Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications in physics and engineering such as strength failure and system collapse. In stress-strength modeling, $R = Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1. The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. **R** can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Other interpretation can be that, the reliability of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables with $TWFr(\lambda_1, b_1, \gamma, \beta)$ and $TWFr(\lambda_2, b_2, \gamma, \beta)$ distributions. The reliability is defined by $R = \int_0^\infty f_1(x, \lambda_1, b_1, \gamma, \beta) F_2(x, \lambda_2, b_2, \gamma, \beta)\, dx$. Then, we can write

$$\boldsymbol{R} = \sum_{i,j=0}^{\infty} a_{i,j} \int_0^\infty h_{b_1 i+j}(x) dx + \sum_{i,j,h,k=0}^{\infty} b_{i,j,k,h} \int_0^\infty h_{b_1 i+j+b_1 h+k}(x) dx$$

Where

$$a_{i,j} = \frac{(-1)^{i+j}}{i!} \binom{-b_1 i}{j} \left(\lambda_1 - 1 - \lambda_1\, 2^i\right)$$

And

$$b_{i,j,k,h} = \frac{(-1)^{i+j+h+k} \left(\lambda_1 - 1 - \lambda_1\, 2^i\right) \binom{-b_1 i}{j} \binom{-b_2 i}{k}}{i!\,(b_1 i + j)[b_1 i + j + b_2 h + k](\lambda_1 - 1 - \lambda_1\, 2^h)^{-1}}$$

Thus, the reliability, R, can be expressed as

$$R = \sum_{i,j=0}^{\infty} a_{i,j} + \sum_{i,j,h,k=0}^{\infty} b_{i,j,k,h}.$$

4.6 Quantile Function

The quantile function (qf) of X, where $X - TWFr(\gamma, \beta, a, b, \lambda)$ is obtained by inverting $X_U = F^{-1}(U)$ as

$$X_U = \gamma \left(-\log \left[1 - \log \left\{ 1 - \frac{(1+\lambda) - \sqrt{(1+\lambda^2) - 4\lambda U}}{2\lambda} \right\}^{-\frac{1}{b}} \right] \right)^{-\frac{1}{\beta}}.$$

Simulating the TWFr random sample is straightforward. If U is a uniform variate on the unit interval (0, 1) then the random variable X follows Eq. (8).
Particularly, the distribution median is

$$X_{0.5} = \gamma \left(-\log \left[1 - \frac{1}{a} \log \left\{ 1 - \frac{(1+\lambda) - \sqrt{(1+\lambda^2) - 4\lambda 0.5}}{2\lambda} \right\}^{-\frac{1}{b}} \right] \right)^{-\frac{1}{\beta}},$$

then

$$X_{0.5} = \gamma \left(-\log \left[1 - \frac{1}{a} \log \left\{ \frac{(\lambda - 1) + \sqrt{(1+\lambda^2)}}{2\lambda} \right\}^{-\frac{1}{b}} \right] \right)^{-\frac{1}{\beta}}.$$

5. Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) of the TWFr distribution are discussed in this section. Then, the log-likelihood function becomes,

$$L = \mathrm{n}\ln\left(ab\beta\alpha^{\beta}\right) - (\beta+1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}$$

$$-(b+1)\sum_{i=1}^{n}\ln\left\{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right\} + \sum_{i=1}^{n}\ln\left\{\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}$$

Differentiate log-likelihood function with respect to parameters.

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \beta\log\gamma - \sum_{i=1}^{n}\frac{2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\log b}{\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}}$$

$$-\sum_{i=1}^{n}\log\left\{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right\} + \sum\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\log b$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + b\log\gamma - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{\gamma}{x_i}\right)^{\beta}\log\beta$$

$$+\sum_{i=1}^{n}\frac{2b\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b-1}\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta}\log\beta}{\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}\left[1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right]^{2}}$$

$$-\sum_{i=1}^{n}\frac{(b+1)\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta}\log\beta}{\left\{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right\}} - b\sum_{i=1}^{n}\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta}\log\beta}{\left[1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right]^{2}}$$

$$\frac{\partial L}{\partial \gamma} = \frac{nb\beta}{\gamma} - \beta\sum_{i=1}^{n}\left(\frac{\gamma}{x_i}\right)^{\beta-1}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{2b\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b-1}\frac{\beta\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta-1}\frac{1}{x_i}}{\left[1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right]^{2}}}{\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}}$$

$$-(b+1)\sum_{i=1}^{n}\frac{\beta\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta-1}\frac{1}{x_i}}{\left\{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right\}}$$

$$+b\sum_{i=1}^{n}\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b-1}\frac{\beta\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\left(\frac{\gamma}{x_i}\right)^{\beta-1}\frac{1}{x_i}}{\left[1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]\right]^{2}}$$

$$\frac{\partial L}{\partial \lambda}=\sum_{i=1}^{n}\frac{2\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}-1}{\left\{1-\lambda+2\lambda\exp\left\{-\left[\frac{\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\gamma}{x_i}\right)^{\beta}\right]}\right]^{b}\right\}\right\}}$$

6. Applications

In this section, we provide two real dataset applications to illustrate the importance of the TWFr distribution. The MLEs of the parameters for these models are calculated and four goodness-of-fit statistics are used to compare the new family with its sub-models. The first data set represents the carbon fibers of 66 observations. The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.8, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.6, 3.65, 3.68, 3.7, 3.75, 4.2, 4.38, 4.42, 4.7, 4.9.

The second data set consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89. These data have also been analyzed by Smith and Naylor (1987).

The MLEs are computed using Mathematica. The goodness of fit measures, including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) statistics are computed to compare the fitted models. Generally the small values of these measures indicate the better the fit to the data. These goodness of fit measures are also computed using Mathematica. We compare the proposed model TWFr, with Kumaraswamy Fréchet (KFr) (Mead, 2014), beta Fréchet (BFr) (Nadarajah & Gupta, 2004), transmuted Fréchet (Mahmoud & Mandouh, 2013), Weibull Fréchet (WFr) (Afify et al. 2016). Their density functions (for x>0) are given by:

WFr: $f(x;\ \alpha,\beta,a,b)=ab\beta\alpha^{\beta}x^{-(\beta+1)}e^{\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]}\left\{1-e^{\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}^{-(b+1)}\exp\left(-a\left[e^{\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}-1\right]^{-b}\right);$

KFr: $f(x;\ \alpha,\beta,a,b)=ab\beta\alpha^{\beta}x^{-(\beta+1)}e^{\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]}\left\{1-e^{\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}^{b-1};$

BFr: $f(x;\ \alpha,\beta,a,b)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{-(\beta+1)}e^{\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]}\left\{1-e^{\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}^{b-1};$

TFr: $f(x;\ \alpha,\beta,b)=\beta\alpha^{\beta}x^{-(\beta+1)}e^{\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\left\{1+b-2be^{\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\};$

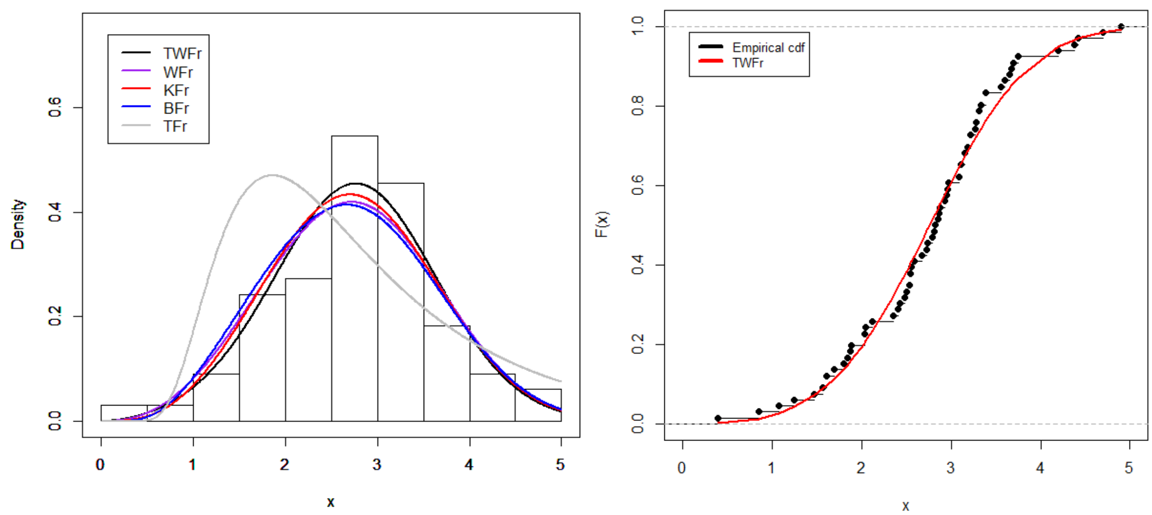
Table 1 and 2 list the ML estimates and model selection statistics of fitted models for the data sets.

Table-1: MLEs and goodness of fit measures for data set 1

Parameters	Probability Distributions				
	KFr	BFr	TFr	WFr	TWFr
α	9.4E+09	522.623	3.53178	0.04630291	1.5069E-05
β	0.12847	0.45492	1.23875	0.674984	0.160797
λ	-	-	-	-	-0.646467
a	1.57844	0.52752	-	1.8525E-06	3.0372E-14
b	2.3E+11	14732.9	2.45	4.72167	16.6086
Log(likelihood)	-86.6727	-87.6214	-90.7381396	-86.2841	-85.35918
A*	0.57577	0.78347	10.0076	0.623812	0.422544
W*	0.1028	0.14884	2.54272	0.118206	0.0704867
AIC	181.543	183.243	187.476	180.68	180.284
BIC	190.104	192.1	194.045	189.327	191.132

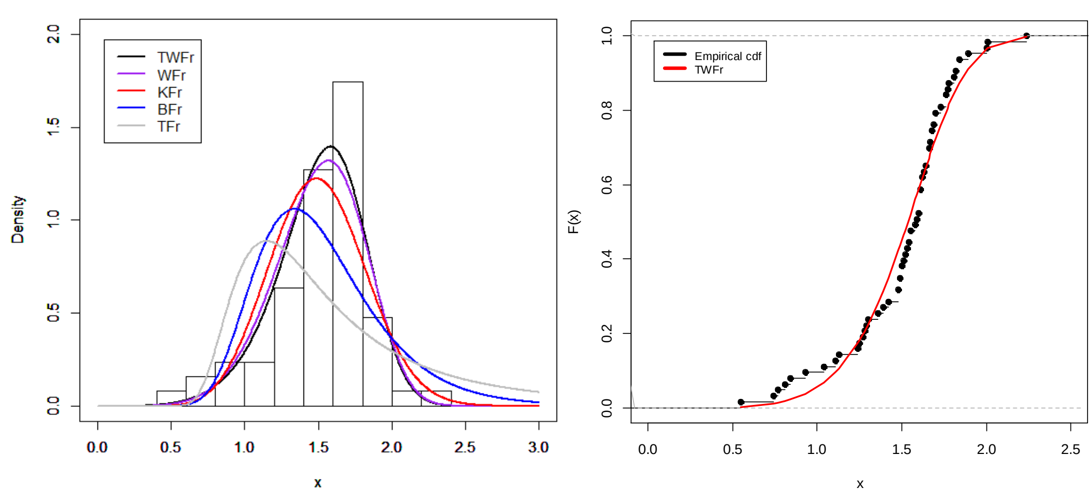
Table-2: MLEs and goodness of fit measures for data set 2

Parameters	Probability distributions					
	KFr	BFr	EFr	TFr	WFr	TWFr
α	2.605E-07	185850000	8.44175	1.0937	0.3865	0.000112174
β	0.481855	0.164219	0.954717	3.22166	0.2436	0.034959799
λ	-	-	-	-0.77447	-	-0.502657301
a	20998	1.93339	132.827	-	1.4762	130.0266213
b	73124.2	2976000000		-	16.8561	105.0252683
Log(likelihood)	-17.6651	-18.6831	-21.999	-43.1516	-15.5005	-14.3672112
A*	1.81873	2.17081	2.79072	6.0437	1.34103	1.05787
W*	0.313038	0.403759	0.510634	1.10797	0.232647	0.17196
AIC	43.3301	45.3661	49.9972	92.3031	39.0011	38.7344
BIC	51.9027	53.9386	56.4266	98.7325	47.536	49.4501



(a) (b)

Fig. 1: Plots of the estimated (a) pdfs and (b) cdf for the TWFr and their sub-models for first dataset-1.



(a) (b)

Fig. 2: Plots of the estimated (a) pdfs and (b) cdf for the TWFr and their sub-models for first dataset-2.

Conclusion

There has been a growing interest among statisticians and applied researchers in developing flexible lifetime models for the betterment of modeling survival data. In this paper, we introduce a new four-parameter extreme value model called the Transmuted Weibull Fréchet (TWFr) distribution, which extends the Fréchet (Fr) distribution. An obvious reason for generalizing Fr distribution is the fact that the generalization provides more flexibility to analyze real life data. We study some of its statistical and mathematical properties. The TWFr density function can be expressed as a linear mixture of Fr densities. We derive explicit expressions for the ordinary and incomplete moments mean deviations, generating function, moments of the residual and reversed residual life. We also obtain the density function of the order statistics and their moments. We estimate the model parameters by maximum likelihood method. The new distribution applied to two real data sets provide better fits than some other related non-nested models. We hope that the proposed model will attract wider applications in areas such as engineering, survival and lifetime data, meteorology, hydrology, economics (income inequality) and others.

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